

# Section 1.2 Row Reduction and Echelon Forms

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Ex: Solve the system of equations

Coefficient matrix  $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} = A$

$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b$

$Ax = b$

$$x_1 - 2x_2 + x_3 = 0 \quad (1)$$

$$2x_2 - 8x_3 = 8 \quad (2)$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \quad (3)$$

augmented matrix

$$\left( \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & 9 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

{ Replant  
Interchange  
scaling

Echelon form

↑  
reduced echelon form



# Echelon Form

- A ~~rectangular~~ matrix is in **echelon form** (or row echelon form) if it has the following three properties:

$$\begin{pmatrix} 3 & -2 & 2 & 1 & 0 \\ 0 & 2 & 4 & -4 & 4 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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  - ▶ All nonzero rows are above any rows of all zeros.
  - ▶ Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  - ▶ All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} 3 & -2 & 2 & 1 & 0 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & 2 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

←  
←  
X

$$\begin{pmatrix} 3 & -2 & 2 & 1 & 0 \\ 0 & 2 & 4 & -4 & 4 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or reduced row echelon form):

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1/3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or reduced row echelon form):
  - ▶ The leading entry in each nonzero row is 1.

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- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

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- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.

- However, the reduced echelon form one obtains from a matrix is unique.

**Theorem (Uniqueness of the Reduced Echelon Form)** Each matrix is row equivalent to one and only one reduced echelon matrix.

$$\left[ \begin{array}{c} * \\ \vdots \\ \end{array} \right] \rightarrow \left[ \begin{array}{c|c} 1 & \dots \\ \vdots & \vdots \\ \hline 0 & \dots \\ \vdots & \vdots \\ \hline 0 & \dots \end{array} \right]$$

- If a matrix  $A$  is row equivalent to an echelon matrix  $U$ , we call  $U$  an **echelon form (or row echelon form) of  $A$** ; if  $U$  is in reduced echelon form, we call  $U$  the **reduced echelon form of  $A$** .

# Pivot Position

- If a matrix  $A$  is row equivalent to an echelon matrix  $U$ , we call  $U$  an **echelon form** (or **row echelon form**) of  $A$ ; if  $U$  is in reduced echelon form, we call  $U$  the **reduced echelon form** of  $A$ .
- A **pivot position** in a matrix  $A$  is a location in  $A$  that corresponds to a leading 1 in the reduced echelon form of  $A$ . A **pivot column** is a column of  $A$  that contains a pivot position.

$$A = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$



$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1/3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot position  
(1,1), (2,2), (3,4)  
Pivot columns:  
 $C_1, C_2, C_4$

- Example: Row reduce the matrix  $A$  below to echelon form, and locate the pivot positions and pivot columns of  $A$ .

$$\begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\begin{array}{l} R_2 + R_1 \\ R_3 + 2R_1 \end{array} \rightarrow \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix} \xrightarrow{\begin{array}{l} \frac{1}{2}R_2 \\ \frac{1}{5}R_3 \end{array}} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

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C1, C2, C4.

$$\begin{array}{l} R_3 - R_2 \\ R_4 + 3R_2 \end{array} \rightarrow \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Row Reduction Algorithm

- Example: Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

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- STEP 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
- STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position. (interchange R1 and R3)
- STEP 3: Use row replacement operations to create zeros in all positions below the pivot.

# Row Reduction Algorithm

- STEP 4: Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

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*echelon form.*

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*reduced echelon form*

- The combination of steps 1–4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

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$$\begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} \textcircled{1} & 0 & -5 & 1 \\ 0 & \textcircled{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \end{cases} \begin{cases} x_1 = 5x_3 + 1 \\ x_2 = -x_3 + 4 \\ x_3 \text{ free} \end{cases}$$

$x_1$   $x_2$

- There are 3 variables because the augmented matrix has four columns.
- The variables  $x_1$  and  $x_2$  corresponding to pivot columns in the matrix are called basic variables. The other variable,  $x_3$ , is called a free variable.

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- In the above example, solve the first and second equations for  $x_1$  and  $x_2$ , we have  $x_1 = 1 + 5x_3$ ,  $x_2 = 4 - x_3$  and  $x_3$  is free.

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- In the above example, solve the first and second equations for  $x_1$  and  $x_2$ , we have  $x_1 = 1 + 5x_3, x_2 = 4 - x_3$  and  $x_3$  is free.
- Each different choice of  $x_3$  determines a (different) solution of the system, and every solution of the system is determined by a choice of  $x_3$ .

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- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

For example, in the above problem, we may take  $x_2$  as free variables, and write  $x_1$  and  $x_3$  in terms of  $x_2$ :

$$x_1 = 21 - 5x_2, x_3 = 4 - x_2 \text{ and } x_2 \text{ is free.}$$

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For example, in the above problem, we may take  $x_2$  as free variables, and write  $x_1$  and  $x_3$  in terms of  $x_2$ :

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- When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

# Existence and Uniqueness Theorem

- **Theorem (Existence and Uniqueness Theorem)** A linear system is consistent **if and only if** the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form  $[0 \dots 0b]$  with  $b$  nonzero.

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- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

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- 3 Continue row reduction to obtain the reduced echelon form.
- 4 Write the system of equations corresponding to the matrix obtained in step 3.
- 5 Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

# Using Row Reduction to Solve a Linear System—Example

Ex: find the general solution of the system

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$

$$-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$$

$$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \xrightarrow[\text{R3}-3\text{R1}]{\text{R2}+2\text{R1}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{R3}+\text{R2}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \leftarrow \text{no solution.}$$

