Linear choosability of sparse graphs

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\textbf{A B S T R A C T}

A linear coloring is a proper coloring such that each pair of color classes induces a union of disjoint paths. We study the linear list chromatic number, denoted $lc_l(G)$, of sparse graphs. The maximum average degree of a graph $G$, denoted $mad(G)$, is the maximum of the average degrees of all subgraphs of $G$. It is clear that any graph $G$ with maximum degree $\Delta(G)$ satisfies $lc_l(G) \geq \lceil \Delta(G)/2 \rceil + 1$. In this paper, we prove the following results: (1) if $mad(G) < 12/5$ and $\Delta(G) \geq 3$, then $lc_l(G) = \lceil \Delta(G)/2 \rceil + 1$, and we give an infinite family of examples to show that this result is best possible; (2) if $mad(G) < 3$ and $\Delta(G) \geq 9$, then $lc_l(G) \leq \lceil \Delta(G)/2 \rceil + 2$, and we give an infinite family of examples to show that the bound on $mad(G)$ cannot be increased in general; (3) if $G$ is planar and has girth at least 5, then $lc_l(G) \leq \lceil \Delta(G)/2 \rceil + 4$.

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\textbf{1. Introduction}

In 1973, Grünbaum introduced \textit{acyclic colorings} [3], which are proper colorings with the additional property that each pair of color classes induces a forest. In 1997, Hind, Molloy, and Reed introduced frugal colorings [4]. A proper coloring is \textit{k-frugal} if the subgraph induced by each pair of color classes has maximum degree less than $k$. Yuster [8] combined the ideas of acyclic coloring and 3-frugal coloring in the notion of a \textit{linear coloring}, which is a proper coloring such that each pair of color classes induces a union of disjoint paths—also called a \textit{linear forest}. We write $lc(G)$ to denote the \textit{linear chromatic number} of $G$, which is the smallest integer $k$ such that $G$ has a proper $k$-coloring in which every pair of color classes induces a linear forest.

We begin by noting easy upper and lower bounds on $lc(G)$. If $G$ is a graph with maximum degree $\Delta(G)$, then we have the naive lower bound $lc(G) \geq \lceil \Delta(G)/2 \rceil + 1$, since each color can appear on at most two neighbors of a vertex of maximum degree. Observe that $lc(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1$, where $\chi(G)$ denotes the chromatic number of $G$ and $G^2$ is the square graph of $G$. Yuster [8] constructed an infinite family of graphs such that $lc(G) \geq C_1 \Delta(G)^{3/2}$, for some constant $C_1$. He also proved an upper bound of $lc(G) \leq C_2 \Delta(G)^{3/2}$, for some constant $C_2$ and for sufficiently large $\Delta(G)$.

Note that trees with maximum degree $\Delta(T)$ have linear chromatic number $\lceil \Delta(T)/2 \rceil + 1$, i.e., the naive lower bound holds with equality (for example, we can color greedily in the order of a breadth-first search from an arbitrary vertex). This equality for trees suggests that sparse graphs might have linear chromatic number close to the naive lower bound. To be more precise: Does there exist a constant $C$ such that every sparse graph $G$ satisfies $lc(G) \leq \lceil \Delta(G)/2 \rceil + C$? To state the previous results related to this question, we first introduce some more notation.

We start with linear list colorings, which are linear colorings from assigned lists. Formally, let $lc_l(G)$ be the \textit{linear list chromatic number} of $G$, that is, the smallest integer $k$ such that if each vertex $v \in V(G)$ is given a list $L(v)$ with $|L(v)| \geq k$,
then $G$ has a linear coloring such that each vertex $v$ gets a color $c(v)$ from its list $L(v)$. When all the lists are the same, linear list coloring is the same as linear coloring. General list coloring was first introduced by Erdős, Rubin, and Taylor [1] and independently by Vizing [7] in the 1970s, and it has been well explored since then [5].

Linear list colorings were first studied by Esperet, Montassier, and Raspaud [2]. The maximum average degree of a graph $G$, denoted $\text{mad}(G)$, is the maximum of the average degrees of all of its subgraphs, i.e., $\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. Observe that the family of all trees is precisely the set of connected graphs with $\text{mad}(G) < 2$ (so indeed we are generalizing our motivating example, trees). The following results were shown in [2]:

**Theorem A** ([2]). Let $G$ be a graph:

1. If $\text{mad}(G) < 8/3$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 3$.
2. If $\text{mad}(G) < 5/2$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 2$.
3. If $\text{mad}(G) < 16/7$ and $\Delta(G) \geq 3$, then $\text{lct}(G) = \lceil \Delta(G)/2 \rceil + 1$.

The girth of a graph $G$, denoted $g(G)$, or simply $g$, is the length of its shortest cycle. By an easy application of Euler’s formula, we see that every planar graph $G$ with girth $g$ satisfies $\text{mad}(G) < 2g/(g - 2)$. So we can obtain some results on planar graphs from the above results. Raspaud and Wang [6] proved somewhat stronger results for planar graphs.

**Theorem B** ([6]). Let $G$ be a planar graph:

1. If $g(G) \geq 5$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 14$.
2. If $g(G) \geq 6$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 4$.
3. If $g(G) \geq 13$ and $\Delta(G) \geq 3$, then $\text{lct}(G) = \lceil \Delta(G)/2 \rceil + 1$.

Our goal in the paper is to improve the results in the above two theorems. We prove the following.

**Theorem 1.** Let $G$ be a graph:

1. If $G$ is planar and has $g(G) \geq 5$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 4$.
2. If $\text{mad}(G) < 3$ and $\Delta(G) \geq 9$, then $\text{lct}(G) \leq \lceil \Delta(G)/2 \rceil + 2$.
3. If $\text{mad}(G) < 12/5$ and $\Delta(G) \geq 3$, then $\text{lct}(G) = \lceil \Delta(G)/2 \rceil + 1$.

Raspaud and Wang [6] conjectured that the bound in Theorem 1(2) holds for all planar graphs with girth at least 6. Since every such graph $G$ has $\text{mad}(G) < 3$, our result proves their conjecture for graphs with $\Delta(G) \geq 9$. Since $\text{mad}(K_{3,3}) = 3$ and $\text{lct}(K_{3,3}) = 5$, we can construct an infinite family of sparse graphs $G$ containing $K_{3,3}$ such that $\text{mad}(G) = 3$, $\Delta(G) = 4$, and $\text{lct}(G) > \lceil \Delta(G)/2 \rceil + 2$. Thus, the maximum degree condition in Theorem 1(2) cannot be lower than 5.

We also note that $\text{lct}(K_{2,3}) = 4 > \lceil \Delta(K_{2,3})/2 \rceil + 1$ and $\text{mad}(K_{2,3}) = 12/5$. Thus, we can construct an infinite family of sparse graphs containing $K_{2,3}$ with maximum degree at most 4. All such graphs have $\text{lct}(G) = \lceil \Delta(G)/2 \rceil + 2$ and can be made sparse enough so that $\text{mad}(G) = \text{mad}(K_{2,3}) = 12/5$. So the bound on $\text{mad}(G)$ in Theorem 1(3) is sharp.

The proofs of our three results all follow the same outline. First, we prove a structural lemma; this says that each graph under consideration must contain at least one from a list of “configurations”. Second, we prove that any minimal counterexample to our theorem cannot contain any of these configurations. In this second step, we begin with a linear list coloring of part of the graph, and show how to extend it to the whole graph. As we extend the coloring, we often say that we “choose $c(v) \in L(v)’$; by this we mean that we pick some color $c(v)$ from $L(v)$ and use $c(v)$ to color vertex $v$. In the following three sections, we will prove our three main results, respectively.

For convenience, we introduce the following notation. A $k$-vertex is a vertex of degree $k$. A $k^+$-vertex ($k^-$-vertex) is a vertex of degree at least (at most) $k$. A $k$-thread is a path of $k + 2$ vertices, where each of the $k$ internal vertices has degree 2, and each of the end vertices has degree at least 3.

### 2. Planar with girth at least 5 implies $\text{lct}(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 4$

**Lemma 1.** If $G$ is a planar graph with $\delta(G) \geq 2$ and with girth at least 5, then $G$ contains one of the following two configurations:

1. A 2-vertex adjacent to a 5$^+$-vertex.
2. A 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Proof.** We use the discharging method, with initial charge $\mu(f) = d(f) - 5$ for each face $f$ and initial charge $\mu(v) = \frac{3}{2}d(v) - 5$ for each vertex $v$. By Euler’s formula, we have $\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = (3|E| - 5|V|) + (2|E| - 5|F|) = -5(|E| - |F|) = -10$. We redistribute charge via the following two discharging rules:

1. Each 4$^+$-vertex $v$ sends charge $\frac{3d(v) - 5}{2d(v)}$ to each incident face (for each time they touch).
2. Each face sends charge 1 to each incident 2-vertex and charge $\frac{1}{6}$ to each incident 3-vertex (for each time they touch).
Theorem 1
Let $M$ be an integer. If $G$ is a planar graph with nonnegative charge, this is a contradiction, since the discharging rules preserve the sum of the charges (which begins negative). We write $\mu^+(v)$ and $\mu^-(v)$ to denote the charge at vertex $v$ or face $f$ after we apply all discharging rules. If $d(v) = 2$, then $\mu^+(v) = \frac{1}{2}(2 - 5) + 2(1) = 0$. If $d(v) = 3$, then $\mu^+(v) = \frac{1}{2}(3 - 5) + 3\left(\frac{1}{3}\right) = 0$. By design, each $4^+$-vertex finishes with charge 0, so we now consider the final charge on each face.

Let $f$ be a face of $G$. For each pair, $u_1$ and $u_2$, of adjacent vertices on $f$, we compute the net charge given from $f$ to $u_1$ and $u_2$. If neither of $u_1$ and $u_2$ is a $2$-vertex, then each vertex receives charge at most $\frac{1}{2}$ from $f$, so the net charge given from $f$ to $u_1$ and $u_2$ is at most $2\left(\frac{1}{2}\right) = 1$. If one of $u_1$ and $u_2$, say $u_1$, is a $2$-vertex, then, since $G$ does not contain (RC1), we have $d(u_2) \geq 6$. Hence, the net charge given from $f$ to $u_1$ and $u_2$ is at most $1 - \frac{2}{3} = \frac{1}{3}$. (This is true because as the degree of a vertex increases beyond 6, the charge it gives to each incident face increases beyond $\frac{1}{3}$.) By a simple counting argument, we see that the net total charge given from $f$ to all incident vertices is at most $\frac{1}{2}(\frac{1}{2}d(f)) = \frac{1}{6}d(f)$. Since $\mu(f) = d(f) - 5$, we see that $\mu^+(f) \geq 0$ when $d(f) \geq 6$. Now we consider 5-faces.

Suppose $f$ is a 5-face. Let $n_2$, $n_3$, and $n_6^+$ denote the number of 2-vertices, 3-vertices, and $6^+$-vertices incident to $f$. Note that $\mu^+(f) \geq -n_2 - \frac{1}{2}n_3 + \frac{3}{2}n_6^+$. From (RC1), we have $n_2 \leq \left\lfloor \frac{d(f)}{2} \right\rfloor = 2$. If $n_2 = 2$, then $n_3 = 0$ and $n_6^+ = 3$, so $\mu^+(v) \geq -2 - \frac{1}{2}(0) + \frac{3}{2}(3) = 0$. If $n_2 = 1$, then $n_6^+ \geq 2$, so $n_3 \leq 2$. Hence, $\mu^+(f) \geq -1 - \frac{1}{2}(2) + \frac{3}{2}(2) = 0$.

Suppose now that $f$ is a 5-face and $n_2 = 0$. Since we have no copy of (RC2), we have either $n_3 = 4$ and $n_6^+ = 1$, or we have $n_3 \leq 3$. In the first case, we get $\mu^+(f) \geq -0 - \frac{1}{6}(4) + \frac{1}{3}(1) = 0$. In the second case, note that $f$ has at least two $4^+$-vertices, each of which gives $f$ charge at least $\frac{1}{4}$. Thus $\mu^+(f) \geq -0 - \frac{1}{6}(3) + \frac{3}{4}(2) = 0$. Hence, every face and every vertex has nonnegative charge. This contradiction completes the proof. □

In Sections 3 and 4, we will only assume bounded maximum degree (rather than planarity and a girth bound).

However, in the proof of the preceding lemma, we needed the stronger hypothesis of planar with girth at least 5. Specifically, we used this hypothesis when considering 5-faces. Our proof relied heavily on the fact that for a 5-face $f$ we have $n_2 \leq \left\lfloor \frac{d(f)}{2} \right\rfloor < \left\lfloor \frac{d(f)}{2} \right\rfloor$.

Now we use Lemma 1 to prove the following linear list coloring result, which immediately implies Theorem 1(1). For technical reasons, we phrase all of our theorems in terms of an integer $M$ such that $\Delta(G) \leq M$. (Without this technical strengthening, when we consider a subgraph $H$ such that $\Delta(H) < \Delta(G)$, we get complications.) Of course, the interesting case is when $M = \Delta(G)$.

Theorem 2. Let $M$ be an integer. If $G$ is a planar graph with $\Delta(G) \leq M$ and girth at least 5, then $\lci(G) \leq \left\lceil \frac{M}{2} \right\rceil + 4$.

Proof. Suppose the theorem is false. Let $G$ be a minimal counterexample and let the list assignment $L$ of size $\left\lceil \frac{M}{2} \right\rceil + 4$ be such that $G$ has no linear list coloring from $L$. Note that $G$ must be connected. Suppose $G$ has a 1-vertex $u$ with neighbor $v$. By minimality, $G - u$ has a linear list coloring from $L$. Let $L'(u)$ denote the list of colors in $L(u)$ that neither appear on $v$, nor appear twice in $N(v)$. Note that $|L'(u)| \geq \left\lceil \frac{M}{2} \right\rceil + 4 - \left(\left\lfloor \frac{M}{2} \right\rfloor + 1\right) = 4$. Thus, if $G$ has a 1-vertex $u$, we can extend a linear list coloring of $G - u$ to $G$. So we may assume that $\delta(G) \geq 2$. Since $G$ is a planar graph with $\delta(G) \geq 2$ and girth at least 5, $G$ contains one of the two configurations specified in Lemma 1.

Case (RC1): First, suppose that $G$ contains a 2-vertex $u$ adjacent to a $5^-$-vertex $v$. Let $w$ be the other neighbor of $u$. By minimality, $G - u$ has a linear list coloring from $L$. In order to avoid creating any 2-colored cycles and to also avoid creating any vertices that have three neighbors with the same color, it is sufficient to avoid coloring $w$ with any color that appears two or more times in $N(v) \cup N(w)$. Furthermore, $u$ must not receive a color used on $v$ or on $w$. Let $L'(u)$ denote the list of colors in $L(u)$ that may still be used on $u$. We have $|L'(u)| \geq \left(\left\lceil \frac{M}{2} \right\rceil + 4 - \left(\left\lfloor \frac{M}{2} \right\rfloor + 1\right) + 1\right) = 4$. Thus, we can extend a linear list coloring of $G - u$ to a linear list coloring of $G$.

Case (RC2): Suppose instead that $G$ contains a 5-face $f$ with four incident 3-vertices and with the fifth incident vertex of degree at most 5. We label the vertices as follows: let $u_1$, $u_2$, $u_3$, and $u_4$ denote successive 3-vertices, and let $v_2$ and $v_3$ denote the neighbors of $u_1$ and $u_3$, not on $f$.

By minimality, $G - \{u_2, u_3\}$ has a linear list coloring from $L$. Now we will extend the coloring to $u_2$ and $u_3$. Let $L'(u_2)$ and $L'(u_3)$ denote the colors in $L(u_2)$ and $L(u_3)$ that are still available for use on $u_2$ and $u_3$. When we color $u_2$, we clearly must avoid the colors on $u_1$ and $v_2$. We also want to avoid creating a 2-colored cycle or a vertex that has three neighbors with the same color. To do this, it suffices to avoid any color that appears on two or more vertices at distance two from $u_2$. This gives us an upper bound on the number of forbidden colors: $2 + \left(\frac{M - 1}{2} + 2\right) = \left\lceil \frac{M}{2} \right\rceil + 3$. So $L'(u_2) = \left\lceil \frac{M}{2} \right\rceil + 4 - (\left\lceil \frac{M}{2} \right\rceil + 3) \geq 1$. An analogous count shows that $|L'(u_3)| \geq 1$. However, we might have $L'(u_2) = L'(u_3)$. Thus, we now refine this argument to show that $|L'(u_2)| \geq 2$ or $|L'(u_3)| \geq 2$.

First suppose that $c(u_1) = c(v_2)$. Since the colors on $u_1$ and $v_2$ are the same, these two vertices only forbid a single color from use on $u_2$, rather than the two colors we accounted for above. Thus we get $|L'(u_2)| \geq 2$. As above, $|L'(u_3)| \geq 1$, so we first color $u_3$, then color $u_2$ with a color not on $u_3$. This gives the desired linear coloring of $G$. Hence, we conclude that $c(u_1) \neq c(v_2)$. 

Since \( c(u_1) \neq c(v_2) \), when we color \( u_3 \), we need not fear creating three neighbors of \( u_2 \) with the same color. Further, we need not worry about giving \( u_3 \) the same color as either \( u_1 \) or \( v_2 \), for the following reason. Any 2-colored cycle that contains \( u_3 \) and either \( u_1 \) or \( v_2 \) must also contain \( u_2 \) and either \( u_4 \) or \( v_3 \). Thus, by requiring that \( u_2 \) does not get a color that appears on two or more vertices at distance two, we avoid such a 2-colored cycle. So in fact, \( u_3 \) only needs to avoid colors that appear on \( v_3 \), on \( u_4 \), or on at least two vertices of \( N(u_3) \cup N(v_3) \). This observation gives us \( |L'(u_3)| \geq \left( \left\lceil \frac{M}{10} \right\rceil + 4 \right) - \left( \left\lceil \frac{M}{10} \right\rceil + 2 \right) = \left( \left\lceil \frac{M}{10} \right\rceil + 2 \right) = 2 \). So we can color \( u_3 \), then color \( u_3 \) with a color not on \( u_2 \). This gives the desired linear list coloring, and completes the proof. \( \square \)

A similar, but more detailed, argument proves that if \( G \) is a planar graph with girth at least 5 and \( \Delta(G) \geq 15 \), then \( lc(G) \leq \left\lceil \frac{2\Delta(G)}{5} \right\rceil + 3 \). A brief sketch of this proof is as follows. First, we can refine Lemma 1 to show that if \( \Delta(G) \geq 15 \), then in (RC2) at most two neighbors of \( u_1, u_2, u_3, \) and \( u_4 \) can have high degree. (The key insight is that our present argument only requires that each \( 6^+ \)-vertex give charge \( \frac{2}{7} \) to each incident face; not charge \( \frac{2}{7}d(v) - 5)/d(v) \). Thus, these high degree vertices have lots of extra charge that they can send to adjacent 3-vertices.) With a more careful analysis, we can show that both the original configuration (RC1) and this strengthened version of (RC2) are reducible even with only \( \left\lceil \frac{\Delta(G)}{5} \right\rceil + 3 \) colors.

### 3. \( mad(G) < 3 \) and \( \Delta(G) \geq 9 \) imply \( lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \)

**Lemma 2.** If \( G \) is a graph with \( mad(G) < 3 \), \( \delta(G) \geq 2 \), and \( \Delta(G) \geq 9 \), then \( G \) contains one of the following five configurations:

(RC1) a 2-vertex \( u \) adjacent to vertices \( v \) and \( w \) such that \( \left\lceil \frac{d(v)}{3} \right\rceil + \left\lceil \frac{d(w)}{3} \right\rceil \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

(RC2) a 3-vertex \( u \) adjacent to a 2-vertex and to two other vertices \( v \) and \( w \), such that \( d(v) + d(w) \leq 8 \).

(RC3) a 3-vertex adjacent to two 2-vertices.

(RC4) a 4-vertex adjacent to four 2-vertices.

(RC5) a 5-vertex \( u \) that is adjacent to four 2-vertices, each of which is adjacent to another \( 8^- \)-vertex; and \( u \) is also adjacent to a fifth \( 3^- \)-vertex.

In fact, the hypothesis \( \Delta(G) \geq 9 \) cannot be omitted (though the lower bound can possibly be reduced), as we show after we prove the lemma.

**Proof.** We use discharging, with initial charge \( \mu(v) = d(v) - 3 \) for each vertex \( v \). Since \( mad(G) < 3 \), the sum of the initial charges is negative. Note that only the 2-vertices have negative charge, so we design our discharging rules to pass charge to the 2-vertices. We redistribute the charge via the following three discharging rules:

(R1) Every 4-vertex gives charge \( \frac{1}{3} \) to each adjacent 2-vertex.

(R2) Every 5-vertex gives charge \( \frac{2}{3} \) to each adjacent 2-vertex that is also adjacent to another \( 8^- \)-vertex, and it gives charge \( \frac{1}{3} \) to every adjacent 3-vertex and every other adjacent 2-vertex.

(R3) Every 6\(^+ \)-vertex \( v \) gives charge \( \frac{d(v) - 3}{d(v)} \) to each adjacent 2-vertex and 3-vertex.

(R4) Every 3-vertex gives its charge (that it received from rules (R2) and (R3)) to its adjacent 2-vertex (if it has one).

We will show that if \( G \) contains none of the five configurations (RC1)-(RC5), then each vertex finishes with nonnegative charge, which is a contradiction. The following observation is an immediate corollary of the fact that \( G \) contains no copy of (RC1). We will use this observation below, to show that every vertex finishes with nonnegative charge.

**Observation 1.** Suppose that a 2-vertex \( u \) has neighbors \( v \) and \( w \).

(i) If \( d(v) \in \{3, 4\} \), then \( d(w) = \Delta(G) \) if \( \Delta(G) \) is odd, and \( d(w) \geq \Delta(G) - 1 \) if \( \Delta(G) \) is even.

(ii) If \( d(v) \in \{5, 6\} \), then \( d(w) \geq \Delta(G) - 2 \) if \( \Delta(G) \) is odd, and \( d(w) \geq \Delta(G) - 3 \) if \( \Delta(G) \) is even.

We now use Observation 1 to show that every vertex finishes with nonnegative charge. It is clear from (R3) that every \( 6^+ \)-vertex finishes with nonnegative charge. The same is true for 3-vertices. So we consider 4-vertices, 5-vertices, and 2-vertices.

Suppose \( d(u) = 4 \). Since \( G \) contains no copy of (RC4), every 4-vertex \( u \) is adjacent to at most three 2-vertices. Thus, we have \( \mu^*(u) \geq \mu(u) - 3(\frac{1}{3}) = 1 - 3(\frac{1}{3}) = 0 \).

Suppose \( d(u) = 5 \). If \( u \) has two or more neighbors that each receive charge at most \( \frac{5}{14} \) from \( u \), then \( \mu^*(u) \geq \mu(u) - 3(\frac{1}{3}) - 2(\frac{5}{14}) = 2 - \frac{14}{7} = 0 \). Similarly, if \( u \) has one neighbor that receives no charge from \( u \), then \( \mu^*(u) \geq \mu(u) - 4(\frac{2}{7}) > 0 \). Hence, we may assume that \( u \) sends charge to each neighbor, and that it sends charge \( \frac{2}{7} \) to at least four of its neighbors. However, this assumption implies that \( G \) contains a copy of configuration (RC5), which is a contradiction.

Finally, suppose \( d(u) = 2 \). Let the neighbors of \( u \) be \( v \) and \( w \). Since \( \mu(u) = -1 \), it suffices to show that \( u \) always receives charge at least 1. If \( d(v) \geq 6 \) and \( d(w) \geq 6 \), then \( v \) and \( w \) each give \( u \) charge at least \( \frac{1}{2} \). So we may assume that \( d(v) \leq 5 \).

Suppose \( d(v) = 5 \). Since \( \Delta(G) \geq 9 \), Observation 1 implies that \( d(w) \geq 7 \). If \( d(w) \in \{7, 8\} \), then \( u \) receives charge at least \( \frac{2}{7} + \frac{3}{7} = 1 \). If \( d(w) \geq 9 \), then \( u \) receives charge at least \( \frac{3}{7} + \frac{3}{7} > 1 \).
If $d(v) = 4$, then Observation 1 implies that $d(w) \geq 9$, so $u$ receives charge at least $\frac{1}{2} + \frac{6}{9} = 1$. If $d(v) = 3$, then the absence of (RC2) implies that at least one neighbor $x$ of $v$ has degree at least 5, so $u$ receives charge at least $\frac{5}{14}$ from $x$. Since $v$ can have at most one adjacent 2-vertex, $u$ gets charge at least $\frac{5}{14}$ from $v$. Hence, the total charge that $u$ receives is at least $\frac{6}{9} + \frac{5}{14} > 1$. □

Now we give two examples to show that the hypothesis $\Delta(G) \geq 9$, in Lemma 2 above, cannot be omitted. (We do suspect, however, that this hypothesis can be replaced by $\Delta(G) \geq 7$, or perhaps even by $\Delta(G) \geq 5$.) We first give an example with maximum degree 3. Let $G$ be the dodecahedron, and let $E$ be a matching in $G$ of size 6, such that every face of $G$ contains one edge of $E$. Form $\overline{G}$ from $G$ by subdividing each edge of the matching. The girth of $\overline{G}$ is 6, so (by an easy application of Euler’s formula), $\text{mad}(\overline{G}) < 3$. Despite having $\text{mad}(\overline{G}) < 3$, $\overline{G}$ does not contain any of the five configurations (RC1)-(RC5) in Lemma 2. Now we give an example with maximum degree 4. Let $G$ be the octahedron, and let $E$ be a perfect matching in $G$. Form $\overline{G}$ from $G$ by subdividing every edge of $G$ except the three edges of $E$. The average degree of $\overline{G}$ is $(4 \times 6 + 2 \times 9)/(6 + 9) = \frac{14}{5}$; it is an easy exercise to verify that $\text{mad}(\overline{G}) = \frac{14}{5}$. Again $\overline{G}$ contains none of the configurations (RC1)-(RC5).

Now we use Lemma 2 to prove the following linear list coloring result, which immediately implies Theorem 1(2).

**Theorem 3.** Let $M \geq 9$ be an integer. If $G$ is a graph with $\text{mad}(G) < 3$ and $\Delta(G) \leq M$, then $\ell_c(G) \leq \left\lceil \frac{M+1}{2} \right\rceil + 2$.

**Proof.** Suppose the theorem is false. Let $G$ be a minimal counterexample and let the list assignment $L$ of size $\left\lceil \frac{M}{2} \right\rceil + 2$ be such that $G$ has no linear list coloring from $L$. Since $M \geq 9$, we have $|L(v)| = \left\lceil \frac{M}{2} \right\rceil + 2 \geq 7$ for every $v \in V$. Note that $G$ must be connected. Suppose $G$ has a 1-vertex $u$ with neighbor $v$. By minimality, $G - u$ has a linear list coloring from $L$. Let $L(u)$ denote the list of colors in $L(u)$ that neither appear on $u$, nor appear twice in $N(u)$. Note that $|L(u)| \geq \left\lceil \frac{M}{2} \right\rceil + 2 = 2$. Thus, if $G$ has a 1-vertex $u$, we can extend a linear list coloring of $G - u$ to $G$. So we may assume that $\delta(G) \geq 2$.

Since $G$ is a graph with $\delta(G) \geq 2$ and $\text{mad}(G) < 3$, $G$ contains one of the five configurations (RC1)-(RC5) specified in Lemma 2. We consider each of these five configurations in turn, and in each case we construct a linear coloring of $G$ from $L$.

**Case (RC1):** Suppose that $G$ contains configuration (RC1). Let $u$ be a 2-vertex adjacent to vertices $v$ and $w$ such that $\left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil < \left\lceil \frac{M}{2} \right\rceil + 2$. By the minimality of $G$, subgraph $G - u$ has a linear list coloring $c$.

If $c(v) \neq c(w)$, then $u$ can receive any color except for $c(v)$, $c(w)$, and those colors that appear twice on $N(v)$ or twice on $N(w)$. The number of colors forbidden is at most $2 + \left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil = \left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil$. Since $|L(u)| = \left\lceil \frac{M}{2} \right\rceil + 2$, and since $\left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil < \left\lceil \frac{M}{2} \right\rceil + 2$, we can extend the coloring to $u$. So we assume instead that $c(v) = c(w) = 1$.

If $c(v) = c(w)$, then (similar to that above), $u$ can receive any color except for $c(v)$ and those colors that appear twice on $N(v)$ and $N(w)$. The number of forbidden colors is at most $1 + \left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil = \left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil$. So, once again, we can extend the coloring to $u$.

**Case (RC2):** Suppose that $G$ contains configuration (RC2). Let $u$ be a 3-vertex adjacent to a 2-vertex and to two other neighbors $v$ and $w$ with $d(v) + d(w) \leq 8$. By the minimality of $G$, subgraph $G - u$ has a linear list coloring from $L$. If all three neighbors of $u$ have the same color, then we will not get a linear coloring of $G$ no matter how we color $u$. In this case, we can recolor the 2-vertex and still have a linear coloring of $G - u$. Now we will extend the coloring to $u$.

Let $L'(u)$ denote the colors in $L(u)$ that are still available for use on $u$. When we color $u$, we clearly must avoid the colors on its three neighbors. We also want to avoid creating a 2-colored cycle or a vertex that has three neighbors with the same color. To do this, it suffices to avoid any color that appears on two or more vertices at distance two from $u$. This gives us an upper bound on the number of forbidden colors: $3 + \left\lceil \frac{d(v) + d(w)}{2} \right\rceil = 3 + \left\lceil \frac{d(v) + d(w)}{2} \right\rceil \leq 3 + \left\lceil \frac{7}{2} \right\rceil = 6$. Since $|L(u)| \geq 7$, we have $|L'(u)| \geq 1$. Thus, we can extend the coloring to $u$.

**Case (RC3):** Suppose that $G$ contains configuration (RC3), shown in Fig. 1. Let $u$ be a 3-vertex that has neighbors $v_1$, $v_2$, and $v_3$ with $d(v_1) = d(v_2) = 2$ and $d(v_3) = 3$. Let $N(v_i) = \{w_i, u\}$ for $i \in \{1, 2\}$. By the minimality of $G$, subgraph $G - \{v_1, v_2, v_3\}$ has a linear list coloring $c$ from $L$. For each uncolored vertex $z \in \{u, v_1, v_2\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on $z$. Note that $|L'(z)| \geq 2$ for each uncolored vertex $z$.

Suppose that $L'(u) = \{c(w_1), c(w_2)\}$; this means that $c(v_3) \notin \{c(w_1), c(w_2)\}$. Color $u$ with $c(w_1)$. Now choose $c(v_1) \in L'(v_1) - c(v_1)$ and $c(v_2) \in L'(v_2) - c(w_1)$. This is a valid linear coloring of $G$.
Suppose instead that $L'(u) \setminus \{c(w_1), c(w_2)\} \neq \emptyset$. Choose $c(u) \in L'(u) \setminus \{c(w_1), c(w_2)\}$, choose $c(v_1) \in L'(v_1) \setminus \{c(u)\}$, and choose $c(v_2) \in L'(v_2) \setminus \{c(u)\}$. This coloring is proper and contains no 2-alternating path through $u$. Hence, it is a linear coloring unless $c(v_1) = c(w_2) = c(v_2)$. If no other choice of $c(v_1)$ and $c(v_2)$ can avoid this problem, then we can conclude that $L'(v_1) = L'(v_2) = \{c(v_1), c_1\}$ (for some color $c_1$); further $L'(u) \setminus \{c(w_1), c(w_2)\} = \{c_1\}$. Suppose we are in this case.

If $c(w_1) = c(w_2)$, then, without loss of generality, $L'(u) = \{c(w_1), c_1\}$. Now let $c(u) = c(w_1)$, $c(v_1) = c_1$, and $c(v_2) = c(w_2)$. This is a valid linear coloring. So, by relabeling, we may assume that $c(w_1) = c_1 = 2$, $c(v_1) = 2$, and $c_1 = 3$. Thus $L'(v_1) = L'(v_2) = \{2, 3\}$ and $L'(u) = \{1, 3\}$.

Note that $\{2, 3\} \subseteq L'(v_i)$ implies that 2 and 3 each appear at most once in $N(w_i)$ (for $i \in \{1, 2\}$). If 3 does not appear on both $N(w_1)$ and $N(w_2)$, then let $c(v_1) = c(v_2) = 3$ and $c(u) = 1$. If 2 does not appear on both $N(w_1)$ and $N(w_2)$, then let $c(u) = 1$, $c(v_1) = 2$, $c(v_2) = 3$ (or $c(u) = 1$, $c(v_1) = 3$, $c(v_2) = 2$). So, we can assume that 2 and 3 each appear once on both $N(w_1)$ and $N(w_2)$. However, now $|L'(u)| \geq (\lceil \frac{m}{2} \rceil + 2) - (\lceil \frac{m+3}{2} \rceil + 1) \geq 3$, which is a contradiction.

Case (RC4): Suppose that $G$ contains configuration (RC4), shown in Fig. 1. Let $u$ be a 4-vertex and let $N(u) = \{v_1 : 1 \leq i \leq 4\}$ such that $d(v_i) = 2$. Also let $N(v_i) = \{u, w_i\}$ for $1 \leq i \leq 4$. By the minimality of $G$, subgraph $G - \{u, v_1, v_2, v_3, v_4\}$ has a linear list coloring from $L$. For each uncolored vertex $z$, let $L'(z)$ denote the list of colors still available for $z$. Note that $|L'(u)| \geq 2$ and $|L'(v_i)| = \lceil \frac{m}{2} \rceil + 2 \geq 7$, since $m \geq 9$.

We can color the $v_i$’s from their lists so that every color is used on at most two $v_i$’s, as follows. If some color $c$ is available for use on two or more $v_i$’s, then use $c$ on exactly two of them, and color each of the remaining $v_i$’s with another color (which could be the same for both of them). Otherwise, all the $v_i$’s have disjoint lists of available colors, so color them arbitrarily.

If the four colors on the $v_i$’s are all distinct, then color $u$ with a fifth color. If $c(v_1) = c(v_2)$ but $c(v_1)$, $c(v_2)$, and $c(v_4)$ are all distinct, then choose $c(u)$ so that $c(u) \not\in \{c(v_1), c(v_2), c(v_4), c(v_3)\}$. Finally, if $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$ (which together imply $c(v_1) \neq c(v_3)$), then choose $c(u)$ so that $c(u) \not\in \{c(v_1), c(v_2), c(v_3), c(v_4)\}$.

Case (RC5): Suppose that $G$ contains configuration (RC5), shown in Fig. 1. Let $u$ be a 5-vertex and let $N(u) = \{v_1 : 1 \leq i \leq 5\}$, such that $d(v_i) = 2$ for $1 \leq i \leq 4$ and $d(v_5) \leq 3$. Also let $N(v_i) = \{u, w_i\}$ for $1 \leq i \leq 4$, where $d(w_i) \leq 8$. By the minimality of $G$, subgraph $G - \{u, v_1, v_2, v_3, v_4\}$ has a linear coloring $c$ from $L$. For each uncolored vertex $z \in \{u, v_1, v_2, v_3, v_4\}$, let $L'(z)$ denote the list of colors still available for $z$. Since $d(w_i) \leq 8$, we have $|L'(v_i)| \geq 3$. Conversely, $|L'(u)| \geq \lceil \frac{m}{2} \rceil + 2 - \lceil \frac{2}{2} \rceil + 1 = \lceil \frac{m}{2} \rceil \geq 5$, since $m \geq 9$. Now let $L'(u) = L'(v_1) = c(v_5)$; note that $|L'(u)| \geq 2$. We now extend the coloring by using the lists $L(u)$ and $L'(v_i)$. We can completely ignore $v_5$ (since we deleted $c(v_5)$ from the lists), so the analysis is exactly the same as in Case (RC4).

As we explained in the Introduction, this theorem immediately yields the following corollary.

**Corollary 1.** If $G$ is a planar, has girth at least 6, and $\Delta(G) \geq 9$, then $\text{lcc}(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 2$.

Although our proof of Theorem 3 relies heavily on the hypothesis $\Delta(G) \geq 9$, we suspect that the Theorem is true even when this hypothesis is removed. Namely, we conjecture that every graph $G$ with $\text{mad}(G) < 3$ satisfies $\text{lcc}(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 2$. If true, this result is best possible, as shown by the graph $K_{3,3}$, since $\text{lcc}(K_{3,3}) = 5$. Furthermore, every graph $G$ with $K_{3,3} \subseteq G$, $\text{mad}(G) = 3$, and $\Delta(G) \in \{3, 4\}$ shows that this result is best possible.

4. $\text{mad}(G) < \frac{12}{5}$ implies $\text{lcc}(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$

In this section, we prove that if $G$ is a graph with $\Delta(G) \geq 3$ and $\text{mad}(G) < \frac{12}{5}$, then $\text{lcc}(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$. For such graphs, we prove an upper bound that matches the trivial lower bound $\text{lcc}(G) \geq \lceil \frac{\Delta(G)}{2} \rceil + 1$. Recall (from the Introduction) that our bound on $\text{mad}(G)$ is best possible, as demonstrated by $K_{2,3}$, since $\text{mad}(K_{2,3}) = \frac{12}{5}$ and $\text{lcc}(K_{2,3}) = \lceil \frac{\Delta(K_{2,3})}{2} \rceil + 1$.

**Lemma 3.** If $G$ is a graph with $\text{mad}(G) < \frac{12}{5}$ and $\Delta(G) \geq 2$, then $G$ contains one of the following four configurations:

1. A 3$^+$-thread,
2. A 3-vertex $v$ incident to two 1$^+$-threads and one 2-thread, such that the vertex at distance two from $v$ along each 1$^+$-thread is a 3$^+$-vertex,
3. Adjacent 3-vertices with at least seven 2-vertices in their incident threads,
4. A path of three vertices $uvw$ with $d(u) = d(w) = d(v) = 3$ such that $w$ is incident to a 2-thread and $u$ and $v$ are each incident to two 2-threads.

**Proof.** We use discharging, with initial charge $\mu(v) = d(v) - \frac{12}{5}$ for each vertex $v$. Since $\text{mad}(G) < \frac{12}{5}$, the sum of the initial charges is negative. We use the following three discharging rules:

1. Every 2-vertex gets charge $\frac{1}{2}$ from each of the endpoints of its thread.
2. Every 3-vertex incident to two 2-threads gets charge $\frac{1}{5}$ from its 3$^+$-neighbor.
3. Every 3-vertex incident to a 1-thread gets charge $\frac{1}{2}$ from the other endpoint of the 1-thread if it is a 4$^+$-vertex.

Now we will show that if $G$ contains none of the configurations (RC1)–(RC4), then every vertex finishes with nonnegative charge, which is a contradiction. If $d(v) = 2$, then $\mu^+(v) = d(v) - \frac{12}{5} + 2 \left(\frac{1}{2}\right) = 0$. If $d(v) \geq 4$, then, since $G$ contains no
Lemma 3

Fig. 2. Configurations (RC2), (RC3), and (RC4) from Lemma 3 and Theorem 4.

$3^+$-threads (by (RC1)), $v$ gives away charge $\frac{1}{2}$ to each of at most $2d(v)$ 2-vertices. Note further that if $v$ gives away charge $\frac{1}{2}$ to $t$ 3-vertices via (R2) and/or (R3), for some constant $t$, then $v$ gives away charge $\frac{1}{2}$ to at most $2d(v) - t$ 2-vertices. Thus, we have $\mu^*(v) \geq \frac{d(v) - \frac{1}{2} - \frac{1}{2}(2d(v))}{\frac{3}{2}(d(v) - 4)} \geq 0$. So we only need to consider 3-vertices.

Let $d(v) = 3$. Suppose $v$ has at most three 2-vertices in its incident threads. If $v$ does not give away charge by (R2), then $v$ gives away charge at most $3(\frac{1}{2})$, so $\mu^*(v) \geq 3 - \frac{3}{2} - 3(\frac{1}{2}) = 0$. If $v$ does give charge by (R2), then, since $G$ contains no copy of (RC3), $v$ has at most two 2-vertices in its incident threads. Thus $v$ gives away charge at most $3(\frac{1}{2})$, unless both $v$ is incident to a 2-thread and also $v$ gives away charge by (R2) to two distinct vertices. However, this situation cannot occur, since it implies that $G$ contains a copy of (RC4), which is a contradiction.

Suppose instead that $v$ has at least four 2-vertices in its incident threads. Since $G$ contains no copy of (RC2), either $v$ is incident to two 2-threads and also adjacent to a $3^+$-vertex, or $v$ is incident to two 1-threads and one 2-thread and the other end of at least one 1-thread is a $4^+$-vertex. In each case, $v$ gives away charge $4(\frac{1}{2})$ and receives charge at least $\frac{1}{2}$, so $\mu^*(v) \geq 3 - \frac{3}{2} - 4(\frac{1}{2}) + \frac{1}{2} = 0$. □

Now we use Lemma 3 to prove the following linear coloring result.

Theorem 4. Let $M \geq 3$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{12}{M}$ and $\Delta(G) \leq M$, then $\text{lcc}(G) = \left\lceil \frac{M}{2} \right\rceil + 1$.

Proof. Suppose the theorem is false. Let $G$ be a minimal counterexample and let list assignment $L$, of size $\left\lceil \frac{M}{2} \right\rceil + 1$, be such that $G$ has no linear list coloring from $L$. Since $M \geq 3$, we have $|L(u)| = \left\lceil \frac{M}{2} \right\rceil + 1 \geq 3$ for all $u \in V$. Note that $G$ must be connected. Suppose that $G$ contains a 1-vertex $u$ with neighbor $v$. By the minimality of $G$, subgraph $G - \{u\}$ has a linear list coloring from $L$. Let $L'(u)$ denote the list of colors in $L(u)$ that neither appear on $v$ nor appear twice in $N(u)$. Note that $|L'(u)| \geq \left(\frac{M}{2}\right) + 1 - \left(\frac{M-1}{2}\right) - 1 \geq 1$. Thus, if $G$ has a 1-vertex $u$, we can extend a linear list coloring of $G - u$ to $G$. So we may assume that $\delta(G) \geq 2$.

Since $G$ has $\delta(G) \geq 2$ and $\text{mad}(G) < \frac{12}{M}$, $G$ contains one of the four configurations specified in Lemma 3. We consider each of these four configurations in turn, and in each case we construct a linear coloring of $G$ from $L$.

Case (RC1): Suppose that $G$ contains (RC1): a $3^+$-thread. Let $u, u_1, u_2, U_3, u_4$ be part of the thread, that is, $d(u_1) \geq 3$, $d(u_2) = d(u_3) = 2$, and $d(u_4) \geq 2$. By the minimality of $G$, subgraph $G - \{u_1\}$ has a linear coloring from $L$. If $c(u_1) = c(u_3)$, then $|L'(u_3)| \geq 2$, so we choose $c(u_2) \in L'(u_2) - \{c(u_3), c(u_1)\}$. If $c(u_1) \neq c(u_3)$, then $|L'(u_3)| \geq 1$, so we choose $c(u_2) \in L'(u_2)$. Note that either $c(u_2) \neq c(u_1)$ or $c(u_2) \neq c(u_3)$, so we have not created a 2-colored cycle.

Case (RC2): Suppose instead that $G$ contains (RC2), shown in Fig. 2. Let $u$ be a $3^+$-thread that is incident to one 2-thread $u, u_1, u_1', u_1''$ with $d(u_1') \geq 3$ and incident to two $1^+$-threads $u, u_2, u_2'$ and $u_3, u_3'$ with $2 \leq d(u_2) \leq 3$ and $2 \leq d(u_3') \leq 3$. By the minimality of $G$, subgraph $G - \{u, u_1, u_2, u_3\}$ has a linear coloring from $L$. Now we will extend the coloring to $G$.

For each uncolored vertex $z \in \{u, u_1, u_2, u_3\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on $z$. When we extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. Note that $|L'(u)| \geq 2$, $|L'(u_2)| \geq 1$, and $|L'(u_3)| \geq 1$.

Suppose $|L(u_2) \cup L'(u_3)| \geq 2$. We choose $c(u_2) \in L'(u_2)$ and $c(u_3) \in L'(u_3)$ such that $c(u_2) \neq c(u_3)$. Next we choose $c(u) \in L'(u) - \{c(u_2), c(u_3)\}$. If $c(u) \neq c(u_2')$, then we choose $c(u_1) \in L'(u_1) - \{c(u)\}$. If instead $c(u) = c(u_2')$, then we choose $c(u_1) \in L'(u_1) - \{c(u_1)\}$. This gives a valid linear coloring.

Suppose instead that $|L(u_2) \cup L'(u_3)| = 1$. Thus $L(u_2) = L'(u_2) = \{a\}$, for some color $a$. Clearly, we must choose $c(u_2) = c(u_3) = a$. Note that this happens only if both $d(u_2) = d(u_3) = 3$ and the two other neighbors of $u_2$ (and $u_3$) have the same color. Now we choose $c(u_1) \in L'(u_1) - \{a, c(u_1)\}$ and $c(u) \in L(u) - \{a\}$.

Since $c(u_1) \neq a$, we have not created any vertex with 3 neighbors of the same color, and we have not created any 2-colored cycle passing through $u_1$. Since $c(u_2)$ does not appear on the other neighbors of $u_2$, we have not created any 2-colored cycle passing through $u_2$.

Case (RC3): Now suppose instead that $G$ contains (RC3): two adjacent 3-vertices with at least seven 2-vertices in their incident threads (shown in Fig. 2). We label the vertices as follows: let $u$ and $v$ be the adjacent 3-vertices, $u$ is incident to two 2-threads $u, u_1, u_1', u_1''$ and $v$ is incident to one 2-thread $v, v_1, v_1', v_2'$. By the minimality of $G$, subgraph $G - \{u, v, u_1, v_1, v_1\}$ has a linear coloring from $L$. Now we will extend the coloring to $G$. For each vertex $z \in \{u, v, u_1, v_1\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on $z$. When we
extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. Note that \(|L'(u_1)| \geq 2, |L'(u_2)| \geq 2, |L'(v_1)| \geq 2, |L'(v)| \geq 3, and |L'(v')| \geq 3; we may assume that equality holds in each case.

Since \(|L'(u)| = 3 > 2 = |L'(u_1)|\), we can choose \(c(u) \in L'(u) - L'(u_1)\). If \(c(u) = c(v_2)\), then choose \(c(v_1) \in L'(v_1) - \{c(u)\}\) and \(c(v) \in L'(v) - \{c(v_1)\}\). If instead \(c(u) \neq c(v_2)\), then choose \(c(u) \in L'(u) - \{c(u)\}\).

Now if \(c(v) \neq c(v_1')\), then choose \(c(v_1') \in L'(v_1) - \{c(v)\}\); if \(c(v) = c(v'_1)\), then choose \(c(v'_1) \in L'(v_1) - \{c(v'_1)\}\). Next, choose \(c(u_1) \in L'(u_1) - \{c(v)\}\). Finally, if \(c(u) = c(u'_2)\), then choose \(c(u_2) \in L'(u_2) - \{c(u'_2)\}\); otherwise, choose \(c(u_2) \in L'(u_2) - \{c(u)\}\).

Recall that \(c(u_1) \neq c(v)\) and either \(c(u) \neq c(v_2)\) or \(c(v_1) \neq c(v_2)\); thus, we do not create any vertices with three neighbors of the same color. By construction, we have no 2-colored cycles through \(u_2\) or \(v_1\). Further, \(c(u_1) \neq c(v)\), so we do not create any 2-colored cycles.

Case (RC4): Suppose that \(G\) contains (RC4). We label the vertices as follows: let \(u, v, w\) be the path; let \(u, u_1, u'_1, u''_1\) and \(u_2, u'_2, u''_2\) be the 2-threads incident to \(u\); let \(v, v_1, v'_1, v''_1\) and \(v_2, v'_2, v''_2\) be the 2-threads incident to \(v\); and let \(w, w_1, w'_1, w''_1\) be the 2-thread incident to \(w\).

By the minimality of \(G\), subgraph \(G - \{u, u_1, u'_1, u''_1, u_2, u'_2, u''_2, v, v_1, v'_1, v''_1, v_2, v'_2, v''_2, w, w_1, w'_1, w''_1\}\) has a linear coloring from \(L\). Now we will extend the coloring to \(G\). For each vertex \(z \in \{u, u_1, u'_1, u''_1, u_2, u'_2, u''_2, v, v_1, v'_1, v''_1, v_2, v'_2, v''_2, w, w_1, w'_1, w''_1\}\), let \(L'(z)\) denote the colors in \(L(z)\) that are still available for use on \(z\). When we extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. We will show explicitly how to color \(u, u_1, u'_1, u''_1, u_2, u'_2, u''_2, v, v_1, v'_1, v''_1, v_2, v'_2, v''_2, w, w_1, w'_1, w''_1\) (and we will color \(v, v_1, v'_1, v''_1, v_2, v'_2, v''_2, w, w_1, w'_1, w''_1\), analogously). We consider two subcases. In fact, we may have one “side” \((u, u_1, u'_1, u''_1, u_2, u'_2, u''_2)\) that is in Subcase (i) and the other side that is in Subcase (ii); this is not a problem, since we color the sides independently.

Subcase (i): Suppose that \(c(u'_1) = c(u''_1)\). If \(c(u'_1) \neq L'(u)\), then we can choose \(c(u_1) \in L'(u_1)\) and \(c(u_2) \in L'(u_2)\) such that \(c(u_1) \neq c(u_2)\), and afterward we choose \(c(u) \in L'(u') - \{c(u'_1), c(u_1), c(u_2)\}\). If \(c(u'_1) \neq L'(u)\), then let \(c(u) = c(u'_1)\). Choose \(c(v)\) analogously. In this instance, we wait to choose \(c(u_1)\) and \(c(u_2)\) until after we choose \(c(u)\).

If \(c(u) = c(v)\), then choose \(c(u_1) \in L'(u_1) - \{c(u)\}\) and \(c(u) \in L'(u) - \{c(u_1), c(u)_2\}\). If \(c(u) \neq c(v)\), then choose \(c(u) \in L'(u) - \{c(u), c(u)_2\}\) and \(c(u_1) \in L'(u_1) - \{c(u), c(u)_2\}\). Finally, choose \(c(u_1) \in L'(u_1) - \{c(u), c(u)_2\}\) if we have not chosen these colors yet; recall that \(c(u) = c(u'_1)\), so \(c(u_1) \neq c(u)\); analogously, \(c(u_2) \neq c(u)\).

Subcase (ii): \(c(u'_1) \neq c(u''_1)\). Choose \(c(u) \in L'(u) - \{c(u'_1), c(u''_1)\}\). Choose \(c(v)\) analogously. Now color \(w\) and \(w_1\) as above. Finally, we color \(u_1, v_1, v_2,\) and \(v_2\) as below.

If we can, we choose \(c(u_1) \in L'(u_1) - \{c(u)\}\) and \(c(u_2) \in L'(u_2) - \{c(u)\}\) such that either \(c(u_1) \neq c(u)\) or \(c(u_2) \neq c(u)\). If this is impossible, then \(L'(u_1) = L'(u_2) = \{c(u), c(u)\}\); furthermore, \(L(u) = \{c(u), c(u'_1), c(u''_1)\}\). Now let \(c(u_1) = c(u_2) = c(u)\) and recolor \(u\) with a new color in \(L'(u) - \{c(u_1), c(u_2), c(u)\}\) (note that \(c(u) \neq L'(u)\)). Finally, color \(v_1, v_2, v_3\) and \(v_4\) analogously.

It is clear that we have created a proper coloring. It is also straightforward to verify that we did not create any vertices with 3 neighbors of the same color, and we did not create any 2-colored cycles. □

This theorem immediately yields the following corollary.

**Corollary 2.** If graph \(G\) is planar with girth at least 12 and \(\Delta(G) \geq 3\), then \(\text{lc}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\).

**References**