On the pagenumber of $k$-trees

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Abstract

A $p$-page embedding of $G$ is a vertex-ordering $\pi$ of $V(G)$ (along the “spine” of a book) and an assignment of edges to $p$ half-planes (called “pages”) such that no page contains crossing edges. The pagenumber of $G$ is the least $p$ such that $G$ has a $p$-page embedding. We disprove a conjecture of Ganley and Heath by showing that for all $k \geq 3$, there are $k$-trees that do not embed in $k$ pages. On the other hand, we present an algorithm that produces $k$-page embeddings for a special class of $k$-trees.

1 Introduction

The pagenumber (or book thickness) of a graph $G$ was introduced by Bernhart and Kainen [1]. Given a graph $G$, a $p$-page embedding of $G$ is a vertex ordering $\pi$ of $V(G)$ (along the “spine” of a book) and an assignment of edges to $p$ half-planes (called “pages”) such that no page contains crossing edges. Equivalently, each page consists of an outerplanar embedding of a subgraph of $G$ having the vertices ordered according to $\pi$ on the unbounded face. These subgraphs decompose $G$. The pagenumber of $G$, denoted $bt(G)$, is the minimum $p$ such that $G$ has a $p$-page embedding. We say that $G$ “embeds in $p$ pages” when $bt(G) \leq p$.

Note that $bt(G) = 1$ if and only if $G$ is outerplanar. Bernhart and Kainen [1] observed that $bt(G) \leq 2$ if and only if $G$ is a subgraph of a Hamiltonian planar graph. Pagenumber has been studied on several classes of graphs, including planar graphs [9], graphs with genus $g$ [5, 6] and complete bipartite graphs [3, 7]. In this paper, we study pagenumber of $k$-trees.

Among several equivalent definitions of $k$-trees, the inductive definition is convenient for our arguments. A $k$-tree is either the complete graph $K_k$ or a graph obtained from a $k$-tree $G$ by adding one vertex whose neighborhood is a $k$-clique in $G$ (a $k$-clique is a set of $k$ pairwise adjacent vertices). The 1-trees are simply the trees, which are outerplanar, and hence they…

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have pagenumber 1. Chung, Leighton, and Rosenberg [2] showed that the pagenumber of every 2-tree is at most 2. Ganley and Heath [4] exhibited k-trees that require k pages and proved that if G is a k-tree, then bt(G) ≤ k + 1. They conjectured that every k-tree embeds in k pages; we disprove this conjecture.

**Theorem 1.** For k ≥ 3, there is a k-tree that does not embed in k pages.

First, we present an algorithm that embeds many k-trees in k pages, using tree-decompositions of graphs. Let G[X] denote the subgraph of G induced by vertex set X. A tree-decomposition of a graph G consists of a host tree T and a family \( \{X_i: i \in V(T)\} \) of subsets of V(G) (called bags, perhaps originally by Bruce Reed) such that (1) \( G = \bigcup_{i \in V(T)} G[X_i] \) and (2) for each \( v \in V(G) \), the set \( \{i: v \in X_i\} \) induces a subtree of T. We use \((T, X)\) to denote a tree-decomposition in which \( X \) is the set of bags.

The width of a tree-decomposition \((T, X)\) is \( \max_{i \in V(T)} \{|X_i| - 1\} \). The treewidth of G is the minimum width among all tree-decompositions of G. (Since every graph has a tree-decomposition with all vertices in one bag, treewidth is well-defined.) A tree-decomposition of width k is smooth if the bags for any two adjacent vertices of the host tree have k common elements. By the inductive definition, a k-tree has a smooth tree-decomposition such that every bag is a (k + 1)-clique.

Togasaki and Yamazaki [8] showed that if G is a k-tree and G has a smooth tree-decomposition whose host tree is a path, then bt(G) ≤ k. We enlarge the family of k-trees for which the conclusion holds.

**Theorem 2.** If a k-tree G has a smooth tree-decomposition with width k such that the host tree has maximum degree at most 3, then bt(G) ≤ k.

The k-tree we construct in Theorem 1 has a smooth tree-decomposition whose host tree has maximum degree k + 2. This leaves open the question of finding the maximum D such that every k-tree having a smooth tree-decomposition whose host tree has maximum degree at most D has a book embedding in k pages. We have shown that 3 ≤ D < k + 2.

# 2 Construction of k-Page Embeddings

We provide an algorithm that produces a k-page embedding of a k-tree G from a smooth tree-decomposition \((T_0, X_0)\) of G in which \( T_0 \) has maximum degree at most 3.

Since the members of \( X_0 \) correspond bijectively to the vertices of \( T_0 \), we refer to the bags as vertices of \( T_0 \). Choose a leaf bag \( \{a_1, \ldots, a_{k+1}\} \) of \( T_0 \); it will be convenient to name this bag \( A_{k+1} \). Note that exactly one vertex of \( A_{k+1} \) does not appear in the neighbor of \( A_{k+1} \) in \( T_0 \); index the elements of \( A_{k+1} \) so that this vertex is \( a_{k+1} \).

In \( T_0 \), each bag X is reached by exactly one path from \( A_{k+1} \). Since \((T_0, X_0)\) is smooth, X contains exactly one vertex that does not appear in any vertex of this path other than X. For each bag \( X_i \), we let \( x_i \) denote this distinguished vertex.

Conversely, since G is connected, every vertex outside \( A_{k+1} \) appears in exactly one closest bag to \( A_{k+1} \) and is the distinguished vertex for that bag. To have every vertex of G be
the distinguished vertex for some bag, we modify $T_0$ by adding a path $(A_1, \ldots, A_k)$ with $A_i = \{a_1, a_2, \ldots, a_i\}$ and $A_k$ adjacent to $A_{k+1}$. Let $T$ denote the new tree, and let $X = X_0 \cup \{A_1, \ldots, A_k\}$; now $(T, X)$ is a tree-decomposition of $G$.

We refer to vertex $A_1$ as the root of $T$. Viewed from $A_1$, the distinguished vertex for each $A_i$ is $a_i$. The new tree-decomposition $(T, X)$ is not smooth, but the $k$ added bags with their distinguished vertices simplify the presentation of the proof. The vertices of $G$ now correspond bijectively to the bags. For $x \in V(G)$, we refer to the bag whose distinguished vertex is $x$ as $\bar{x}$; when the context is clear we write $X$ for $\bar{x}$.

While exploring $T$ from the root, the algorithm uses this bijection from $V(G)$ to $V(T)$ to produce a vertex ordering and a $k$-edge-coloring of $G$ so that the endpoints of two edges with the same color do not occur alternately in the vertex ordering. Such an ordering and coloring define a $k$-page embedding. The idea is to use the correspondence between vertices and bags to color the edges of $T$ using $k + 1$ colors, and then use the edge-coloring of $T$ to produce the $k$-edge-coloring of $G$.

In a graph, a $u, v$-path is a path from $u$ to $v$. We say that $X$ is an ancestor of $Y$ and $Y$ is a descendant of $X$ if $X$ lies on the $A_1, Y$-path in $T$. We will use the following statement about the relationship between $G$ and $T$ to define the edge-coloring of $G$.

**Lemma 3.** If $xy \in E(G)$, then $X$ is an ancestor of $Y$ or $Y$ is an ancestor of $X$ in $T$.

**Proof.** If $xy \in E(G)$, then $x$ and $y$ must appear in some common bag; since the bags containing a vertex of $G$ induce a subtree of $T$, every bag in the $X, Y$-path in $T$ contains $x$ or $y$. Note also that $x$ does not appear in any bag that is an ancestor of $X$ in the rooted tree $T$. The claim follows. \hfill $\square$

We refer to the subtrees of $T$ rooted at the left and right children of $X$ as the (left and right) subtrees of $X$.

### 2.1 The algorithm

First we produce the vertex ordering $\pi$ from $T$. Initialize $\pi$ to $(a_1)$. Begin a breadth-first search of $T$ from bag $A_1$. Designate the child(ren) of a bag $X$ in $T$ as its left-child or right-child, arbitrarily. When searching from bag $X$, having already assigned vertex $x$ a position in $\pi$, place the vertex corresponding to its left child (if it has one) immediately before $x$ in $\pi$ and the vertex corresponding to its right child (if it has one) immediately after $x$ in $\pi$. The vertices for bags in the left subtree of $X$ comprise a consecutive segment immediately before $x$ under $\pi$, and those corresponding to the right subtree of $X$ comprise a consecutive segment immediately after $x$ under $\pi$.

For a bag $Y \in V(T) - \{A_1, \ldots, A_{k+1}\}$ with parent $X$, recall that $|X - Y| = 1$ and that $\overline{X - Y}$ denotes the bag associated with the vertex of $X - Y$. When $Z$ is an ancestor of $Y$, we use $Z : Y$ to denote the edge incident to $Z$ on the $Z, Y$-path in $T$.

Define a $(k + 1)$-coloring $f$ of $E(T)$ as follows. For each edge in $T$, one endpoint is the parent of the other. When $X$ is the parent of $Y$ in $T$, let
contradicts the choice of $x$.

The definition of $x$ reappears. Since $x$ otherwise appears again on the initial edge $xy$, we may assume by symmetry that $X$ is an ancestor of $Y$. Define $g(xy) = f(X : Y)$.

2.2 Validity of the algorithm

First we show that $g$ uses only the colors 1 through $k$.

**Lemma 4.** No edge in $G$ is assigned color $k + 1$ under $g$.

**Proof.** The color $g(xy)$ is the color on an edge in $T$. Since $g(xy) = f(X : Y)$, we have $g(xy) = f(XZ)$, where $Z$ is the child of $X$ on the $X,Y$-path in $T$. If $f(XZ) = k + 1$, then the definition of $f$ implies that $x$ appears in no bag in the subtree of $X$ that contains $Z$, and thus $x$ and $y$ could not appear in a bag together and could not form an edge.

For colors other than $k + 1$, we think of the color on an edge from $X$ to a child of it in $T$ as the color associated with $x$ in the subtree rooted at that child. For such an edge $XY$, let $w$ be the unique vertex of $X - Y$. When $f(XY) \neq k + 1$, the value $f(XY)$ is the color associated with $w$ in the subtree of $W$ that contains $XY$, by the definition of $f$.

**Lemma 5.** If $X$ is an ancestor of $Y$ such that $x \in Y$, then the color $j$ associated with $x$ in the subtree of $X$ that contains $Y$ does not appear on any edge of the $X,Y$-path in $T$ except the initial edge $X : Y$.

**Proof.** Consider a bag $X$ closest to $A_1$ in $T$ at which the claim fails. We have $j \leq k$, since otherwise $x \notin Y$, as observed in the proof of Lemma 4. Note that $j = f(X : Y)$. If $j$ appears again on the $X,Y$-path, then let $ZZ'$ with parent $Z$ be the edge on which it first reappears. Since $j$ reappears on $ZZ'$, the vertex $Z$ cannot be $A_j$. Hence the definition of $f$ yields $f(ZZ') = f(W : Z')$, where $\{w\} = Z - Z'$. Hence $w \notin Y$; since $x \in Y$, we have $x \neq w$. We conclude that $W$ is an ancestor of $X$, since $ZZ'$ was the first reappearance of $j$. Now $j$ is the color associated with $w$ in the subtree of $W$ that contains $Z$, and $w \in Z$. This contradicts the choice of $X$ as the failure closest to $A_1$.

**Proof of Theorem 2.** By Lemma 4, $g$ is a $k$-edge-coloring of $G$. It remains to show that $g$ does not give the same color to edges whose endpoints alternate in $\pi$. Let $xy$ and $uv$ be such edges. By Lemma 3, we may assume that $X$ is an ancestor of $Y$ and $U$ is an ancestor of $V$. Since the algorithm is symmetric with respect to left and right, we may also assume that $Y$ is in the right subtree of $X$, and hence $\pi(x) < \pi(y)$. Recall that $g(xy) = f(X : Y)$.

We show that $g(uv) \neq g(xy)$. Since the right subtree of $X$ is listed immediately after $X$ under $\pi$ and the edge $uv$ crosses the edge $xy$, the right subtree of $X$ must contain $U$ or $V$.
Suppose first that $U$ is in the right subtree of $X$. This implies that $V$ is also in the right subtree of $X$, since $U$ is an ancestor of $V$.

If $V$ is in the left subtree of $U$, then $\pi(x) < \pi(v) < \pi(y) < \pi(u)$. Since the vertices of this subtree appear just before $U$ in the ordering, $Y$ also must be in the left subtree of $U$. Thus $U$ lies along the $X,Y$-path in $T$, and by Lemma 5 the color $g(xy)$ associated with $X$ in its right subtree cannot be the same as the color $g(uv)$ associated with $U$ in its left subtree.

On the other hand, if $V$ is in the right subtree of $U$, then $\pi(x) < \pi(u) < \pi(y) < \pi(v)$, and we see that $Y$ is also in the right subtree of $U$. Again, $U$ lies along the $X,Y$-path in $T$, and Lemma 5 again yields $g(uv) \neq g(xy)$.

Finally, if $U$ is not in the right subtree of $X$, then $V$ must be. Since $U$ is an ancestor of $V$ but is not in the right subtree of $X$, it must be an ancestor of $X$. Now $X$ lies along the $U,V$-path in $T$. By Lemma 5, we conclude that $g(uv) \neq g(xy)$. Therefore, our coloring $g$ together with our ordering $\pi$ yields a valid book embedding of $G$ in $k$ pages. \hfill \Box

Given the smooth tree-decomposition used by the algorithm, the computations by which the algorithm produces the $k$-page embedding can easily be implemented to run in constant time per edge. Since $k$ is fixed, this is linear in the number of vertices.

3 A $k$-Tree With No $k$-Page Embedding

We construct a $k$-tree $G$ that does not embed in $k$ pages. Given any ordering of $V(G)$, we use pigeonholing arguments to produce an induced subgraph of $G$ that cannot be embedded in $k$ pages under that ordering. This suffices, since a $k$-page embedding of $G$ contains a $k$-page embedding of every induced subgraph.

The graph $G$ has a central $k$-clique $X$ with vertices $x_1, \ldots, x_k$. Next we add vertices $y_1, \ldots, y_{kN}$, where $N = (k^2 + k + 5)$, each adjacent to all of $X$. Finally, we add many vertices, called children, each adjacent to $k-1$ vertices in $X$ and one $y_i$. A child has type $(i,j)$ if it is adjacent to $y_i$ and nonadjacent to $x_j$. There are $k^2N$ different types of children. We create $3k(Nk + k + N)$ children of each type, so $G$ altogether has $3k^3N(Nk + k + N)$ children. We refer to all children adjacent to vertex $x_i$ (or $y_i$) as the children of $x_i$ (or $y_i$).

Fix a circular ordering $\pi$ of $V(G)$; we will show that $G$ has no $k$-page embedding under $\pi$. By the Pigeonhole Principle, there are at least $N$ vertices of $\{y_1, \ldots, y_{kN}\}$ between some two vertices of $X$. Hence we may assume by relabeling that $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_N$ appear in that order in $\pi$, with their children somehow interspersed. We delete the remaining vertices of $y_1, \ldots, y_{kN}$ and all their children to obtain an induced subgraph $G_1$. Let $Y = \{y_1, \ldots, y_N\}$, and call $X \cup Y$ the parents. Two vertices $u$ and $v$ are the endpoints of two segments in $\pi$. Sometimes one of those segments does not have internal vertices from both $X$ and $Y$; in this case we refer to those internal vertices as the vertices between $u$ and $v$.

**Lemma 6.** Within $\pi$, there is a subordering consisting of $X \cup Y$ and $3k$ children of each type in $G_1$, such that the children of any type appear consecutively.

*Proof.* We iteratively select $3k$ children of some type, until we obtain all the types. Starting from a vertex $a$ (say $a = x_1$, for example), a step ends when we reach a parent vertex or
obtain $3k$ children of the same unselected type. In the latter case, select these $3k$ vertices. In either case, let the last vertex reached be $a$ and continue.

We claim that all types are selected by the time we return to $x_1$. Suppose that a particular type is not selected. In each step, we see at most $3k-1$ vertices of that type. The number of steps is $r+k+N$, where $r$ is the number of types selected. Since there are $3k(Nk+k+N)$ children of each type, we must have selected children of all $Nk$ types.

Let $G_2$ be the subgraph of $G_1$ induced by the parents and the children selected in Lemma 6. We will show that $G_2$ does not embed in $k$ pages under $\pi$. As we discard vertices to study smaller subgraphs, we refer to the ordering of the remaining vertices within $\pi$ when we say that the induced subgraph has no $k$-page embedding under $\pi$.

We say that vertices $a_1, \ldots, a_m$ form a twist of size $m$ with $b_1, \ldots, b_m$ if $a_1, \ldots, a_m, b_1, \ldots, b_m$ appear in that order in $\pi$ and $a_i$ and $b_i$ are adjacent for $1 \leq i \leq m$. Note that if a vertex ordering contains a twist of size $m$, then every book embedding using that ordering requires at least $m$ pages, as there are $m$ pairwise intersecting edges induced by the vertices of the twist that require distinct pages.

A set $Z$ of children of the same type have the same neighborhood in $G$. In a $k$-page embedding of $G_2$, we say that the vertices of $Z$ have the same edge assignment if for every neighbor $v$ of the vertices in $Z$, the edges from $v$ to $Z$ lie on the same page. We use $N(v)$ for the set of neighbors of vertex $v$ in $G$.

**Lemma 7.** In a $k$-page embedding of $G_2$ under $\pi$, the central $k$ children of any one type have the same edge assignment.

**Proof.** Let $z$ be a child of type $(i,j)$, and let $v_1, \ldots, v_k$ be the neighbors of $z$ in order of their appearance in $\pi$. Group the $3k$ consecutive children of type $(i,j)$ into three runs $A, B, C$ of size $k$. For $v_r \in N(z)$, we show that all edges from $v_r$ to $B$ lie on the same page.

Fix vertices $a_1, \ldots, a_{r-1}$ in $A$ and $c_{r+1}, \ldots, c_k$ in $C$. Given $z' \in B$, note that the vertices $a_1, \ldots, a_{r-1}, z', c_{r+1}, \ldots, c_k$ form a twist of size $k$ with $v_1, \ldots, v_k$. Since $a_1, \ldots, a_{r-1}$ and $c_{r+1}, \ldots, c_k$ are fixed, only the edge from $v_r$ to a vertex of $B$ varies, and it must avoid the $k-1$ pages of the other edges in the twist. Hence all edges from $v_r$ to $B$ lie on the same page. Since this holds for all $r$, the vertices of $B$ have the same edge assignment.

Let $G_3$ be the subgraph of $G_2$ induced by the parents and the $k$ central children of each type. In fact, we will further restrict the vertex set by keeping only five vertices of $Y$ and their children, along with $X$. The next simple observation using twists enables us to select a few special vertices of $Y$.

**Lemma 8.** Let $x_0 = y_N$ and $x_{k+1} = y_1$. In a $k$-page embedding of $G_3$ under $\pi$, for every $j$ with $0 \leq j \leq k$, at most $k$ vertices of $Y$ have children between $x_j$ and $x_{j+1}$.

**Proof.** Suppose that $\{y_{i_1}, \ldots, y_{i_{k+1}}\}$ have children between $x_j$ and $x_{j+1}$, with $i_1 < \cdots < i_{k+1}$, and let $z$ be a child of $y_{i_{j+1}}$ between $x_j$ and $x_{j+1}$. Now $y_{i_1}, \ldots, y_{i_{k+1}}$ form a twist of size $k+1$ with $x_1, x_2, \ldots, x_j, z, x_{j+1}, \ldots, x_k$, preventing $G_3$ from embedding in $k$ pages.

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In Lemma 7, we proved that in a \( k \)-page embedding of \( G_3 \) under \( \pi \), the children of any one type have the same edge assignment (and appear consecutively). By Lemma 8, at most \( k(k + 1) \) vertices of \( Y \) have children (in \( G_3 \)) along the part of the circle from \( y_N \) to \( y_1 \) that contains \( X \). Since \( N = k^2 + k + 5 = k(k + 1) + 5 \), at least five vertices of \( Y \) have all their children (all \( k \) types) along the part of the circle from \( y_1 \) to \( y_N \).

In particular, there are at least three such vertices of \( Y \) aside from \( y_1 \) and \( y_N \). Let \( y_a, y_b, y_c \) be three such vertices, with \( a < b < c \). Let \( Z_{i,j} \) denote the set of \( k \) children of type \((i, j)\) in \( G_3 \), and let \( Z = \bigcup_{(i,j) \in \{a,b,c\} \times [k]} Z_{i,j} \). Let \( G_4 \) be the subgraph of \( G_3 \) induced by \( X \cup \{y_1, y_a, y_b, y_c, y_N\} \cup Z \). It suffices to show that \( G_4 \) does not embed in \( k \) pages under \( \pi \).

Assume henceforth that we have a \( k \)-page embedding of \( G_4 \) under \( \pi \).

The sets \( Z_{i,j} \) for \( j \in [k] \) and \( i \in \{a, b, c\} \) are located along the part of the circle from \( y_1 \) to \( y_N \) that avoids \( X \). We say that \( Z_{i,r} \) is before \( Z_{i,s} \) if it is encountered first when following this part of the circle from \( y_1 \) to \( y_N \) (similarly define \( \text{after} \)).

**Lemma 9.** For \( r < s \), if \( Z_{i,r} \) and \( Z_{i,s} \) are on the same side of \( y_i \) (both before \( y_i \) or both after \( y_i \)), then \( Z_{i,r} \) is before \( Z_{i,s} \).

**Proof.** We state the proof for when \( Z_{i,r} \) and \( Z_{i,s} \) are both before \( y_i \); the other argument is symmetric. Suppose that \( Z_{i,s} \) is before \( Z_{i,r} \). Since \( s \in [k] \), we may choose \( S \subseteq Z_{i,s} \) and \( R \subseteq Z_{i,r} \) with \(|S| = s\) and \(|R| = k + 1 - s\). Since the vertices of \( Z_{i,j} \) are adjacent to all of \( X - \{x_j\} \), we have \( S \subseteq N(x_r) \) and \( R \subseteq N(x_s) \). We conclude that \( y_i, x_1, \ldots, x_k \) form a twist of size \( k + 1 \) with the vertices of \( S \cup R \).

The earlier children of \( y_i \) are those before \( y_i \); the others are its later children.

**Lemma 10.** All edges joining \( y_i \) to its earlier children lie on the same page. Symmetrically, those joining \( y_i \) to its later children lie on the same page.

**Proof.** Consider the earlier children of \( y_i \). By Lemma 7, the vertices of a set \( Z_{i,j} \) have the same edge assignment. Hence it suffices to show that an edge from \( y_i \) to \( Z_{i,r} \) and an edge from \( y_i \) to \( Z_{i,s} \) are on the same page.

We may assume that \( Z_{i,r} \) is before \( Z_{i,s} \). Choose \( w \in Z_{i,r} \), and let \( z \) be the first vertex of \( Z_{i,s} \). We have picked \( z \) so that all edges from \( X \) to the rest of \( Z_{i,s} \) cross \( y_i z \) (and also \( y_i w \)). The \( k - 1 \) vertices of \( Z_{i,s} - \{z\} \) form a twist with the \( k - 1 \) vertices of \( X - \{x_s\} \). Therefore, only one page remains for \( y_i z \) and \( y_i w \).

**Lemma 11.** If \( x_1, \ldots, x_k \) form twists with both \( v_1, \ldots, v_k \) and \( w_1, \ldots, w_k \), where \( v_1, \ldots, v_k \) come before \( w_1, \ldots, w_k \) except possibly \( v_k = w_1 \), then for \( 1 \leq r \leq k \) the edges incident to \( x_r \) in the two twists are on the same page.

**Proof.** Observe that \( x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k \) form a twist with \( v_1, \ldots, v_{r-1}, w_{r+1}, \ldots, w_k \). The edges \( x_r v_r \) and \( x_r w_r \) cross all \( k - 1 \) edges formed by the twist.

**Lemma 12.** If \( Z_{i,1} \) is before \( Z_{i,k} \) for some \( i \) in \( \{a, b, c\} \), then \( G_4 \) does not embed in \( k \) pages under \( \pi \).
Proof. The vertices of $X$ form twists with both $\{y_1\} \cup Z_{i,1}$ and $Z_{i,k} \cup \{y_N\}$. By Lemma 11, the edges incident to $x_r$ in the two twists are on the same page, which we call page $r$, for $1 \leq r \leq k$. By Lemma 7, the edges from $x_r$ to all of $Z_{i,1} \cup Z_{i,k}$ are on the same page.

Suppose that some $Z_{i,j}$ lies after $Z_{i,1}$ and before $Z_{i,k}$. Any edge from $x_r$ to $Z_{i,j}$ crosses the edges from $x_1, \ldots, x_{r-1}$ to $\{y_1\} \cup Z_{i,1}$ and from $x_{r+1}, \ldots, x_k$ to $Z_{i,k} \cup \{y_N\}$. Therefore, all edges from $x_r$ to $Z_{i,j}$ lie on page $r$.

Since $Z_{i,1}$ is before $Z_{i,k}$, it follows that $Z_{i,1}$ is before $y_i$ or $Z_{i,k}$ is after $y_i$. If both, then since $k \geq 3$, some $Z_{i,j}$ is after $Z_{i,1}$ and before $Z_{i,k}$. If $Z_{i,j}$ is before $y_i$, then $Z_{i,1}$ and $Z_{i,j}$ are before $y_i$; otherwise, $Z_{i,k}$ and $Z_{i,j}$ are after $y_i$. By symmetry, we may assume the former.

Let $z$ be the first vertex of $Z_{i,j}$. Since $y_i z$ crosses the edges from $X - \{x_j\}$ to the last vertex of $Z_{i,j}$, edge $y_i z$ lies on page $j$. Let $z'$ be the first vertex of $Z_{i,1}$. Since $y_i z'$ crosses the edges from $X - \{x_1\}$ to the last vertex of $Z_{i,1}$, edge $y_i z'$ lies on page 1. However, since $j \neq 1$, this contradicts Lemma 10. We conclude that $G_4$ does not embed in $k$ pages under $\pi$. □

Lemma 13. If $Z_{i,k}$ is before $Z_{i,1}$ for all $i \in \{a, b, c\}$, then $G_4$ does not embed in $k$ pages under $\pi$.

Proof. For $i \in \{a, b, c\}$, by Lemma 9, $y_i$ is after $Z_{i,k}$ and before $Z_{i,1}$. Since $k \geq 3$, we may choose $j \in [k] - \{1, k\}$. Now $Z_{b,j}$ occurs before or after $y_b$; by symmetry, we may assume that $Z_{b,j}$ is before $y_b$ (hence also before $Z_{b,k}$, by Lemma 9). Now consider the location of $y_a$.

Case 1: $y_a$ is after some child of $y_b$ (on the left in Fig. 1). Let $Z_{b,r}$ be the last $k$ children of $y_b$ before $y_a$. Note that $r > 1$. Now $y_b, x_1, \ldots, x_k$ form a twist of size $k + 1$ with $r$ vertices of $Z_{b,r}, y_a$, and $k - r$ vertices of $Z_{a,1}$ ($Z_{a,1}$ is after $y_a$ by Lemma 9; this contribution is empty if $r = k$). Hence in this case $G_4$ does not embed in $k$ pages under $\pi$.

Case 2: $y_a$ is before all children of $y_b$ (on the right in Fig. 1). Thus $y_a$ is before $Z_{b,j}$, and $Z_{a,k}$ is before $y_a$. Since $j < k$, vertices $x_1, \ldots, x_k$ form a twist with $k - 1$ vertices of $Z_{a,k}$ and
the last vertex of $Z_{b,j}$ (call it $z$). Also recall that $x_1, \ldots, x_k$ form a twist with $\{y_b\} \cup Z_{b,1}$. By Lemma 11, $x_kz$ and $x_kw$ lie on the same page, where $w$ is the last vertex of $Z_{b,1}$.

Let $w'$ be the first vertex of $Z_{b,k}$. Note that $x_1, \ldots, x_k$ form a twist with $(Z_{b,k} - \{w'\}) \cup \{w\}$. Since $y_bw'$ crosses its $k - 1$ edges other than $x_kw$, edges $y_bw'$ and $x_kw$ lie on the same page.

Finally, by Lemma 10, $y_bw'$ lies on the same page with $y_bz'$, where $z'$ is the first vertex of $Z_{b,j}$. Now $y_bz'$ and $x_kz$ lie on the same page, but they cross. Hence in this case also $G_4$ does not embed in $k$ pages under $\pi$.

Lemmas 12 and 13 eliminate all possibilities for $k$-page embeddings and complete the proof of the theorem.

Finally, we remark that the $k$-tree $G$ constructed for the proof of Theorem 1 has a smooth tree-decomposition with a host tree of maximum degree $k + 2$. Let $X_i = X \cup \{y_i\}$ for $1 \leq i \leq kN$. Form a path with vertices $X_1, \ldots, X_{kN}$. For each $X_i$ and $x_j$, form a path with endpoint $X_i$ whose vertices correspond to bags formed by adding to $X_i - \{x_j\}$ one child of type $(i, j)$. This is the desired tree-decomposition of $G$. As mentioned in the introduction, this leaves the question of what is the largest degree of host trees in tree-decompositions of $k$-trees that guarantees the existence of a $k$-page embedding.

References


