Math 432 lec 17  \( k \)-connected graphs

A subdivision of any 2-connected graph is still 2-connected.

Theorem (ear-decomposition) A graph is 2-connected if and only if it has a ear-decomposition. In addition, any cycle can be taken as the initial cycle of some ear-decomposition.

Menger’s Theorem (global version): a graph is \( k \)-connected if and only if there are \( k \) internally disjoint paths between any pair of vertices.

Proof: induction on \( d(u, v) \).

Menger’s Theorem (local version): the maximum number of internally disjoint paths between \( x \) and \( y \) equals to the minimum number of vertices in a vertex cut separating \( x \) and \( y \). \( (\kappa(x, y) = \lambda(x, y)) \)

Proof: use induction on \( n \) and consider two cases depending on whether a \( x, y \)-cut contains \( N(x) \) or \( N(y) \).

Applications of Menger’s Theorem:

Fan Lemma: A graph is \( k \)-connected if and only if it has at least \( k + 1 \) vertices and, for every choice of \( x, U \) with \( |U| \geq k \), it has a \( x, U \)-fan of size \( k \), where an \( x, U \)-fan is set of \( x, U \)-paths such that any two of them share only the vertex \( x \).

Theorem: Use local version of Menger’s Theorem to prove the Konig-Egervary Theorem.

Let \( A = \{A_1, A_2, \ldots, A_m\} \) and \( B = \{B_1, B_2, \ldots, B_m\} \) be two system of sets taking from \([n]\). We may ask when there exists a common system of distinct representatives (CSDR), meaning a set of \( m \) elements that is an SDR for \( A \) and also for \( B \).

Theorem (Ford-Fulkerson, 1958) Family \( A = \{A_1, A_2, \ldots, A_m\} \) and \( B = \{B_1, B_2, \ldots, B_m\} \) have a common system of distinct representatives (CSDR) if and only if

\[
|\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)| \geq |I| + |J| - m
\]

for each pair \( I, J \subseteq [m] \).

Proof: construct a graph with vertices \( s, t, a_i, b_j \) \((1 \leq i, j \leq m)\) and numbers in \( A_i \)'s and \( B_j \)'s and edges \( \{sa_i, b_jt\} \cup \{a_i x : x \in A_i\} \cup b_jy : y \in B_j\} \). A \( s, t \)-path uses a common neighbor of \( A_i \) and \( B_j \), and we need \( m \) disjoint \( s, t \)-paths. So by Menger’s theorem, we just need to show that any \( s, t \)-cut has size at least \( m \). We then consider any \( s, t \)-cut \( R \).

Example: Let \( A = \{12, 23, 31\} \) and \( B = \{14, 24, 1234\} \).