Section 1.4 Matrix equation (Cont.) and Section 1.5 Solution sets of linear systems

Gexin Yu
gyu@wm.edu

College of William and Mary

Example: computing Ax

- Ex: compute Ax, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- Solution: By definition,

$$\begin{aligned}
\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}
\end{aligned}$$

Equivalent formulations (review)

- Theorem 3: if A is an $m \times n$ matrix, with column vectors $a_1, a_2, \ldots, a_n$, and if b is a vector in $\mathbf{R}^m$, then the following three equations have the same solution set

  1. matrix equation $Ax = b$
  2. vector equation $x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = b$
  3. the linear system with augmented matrix

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}$$

- So the equation $Ax = b$ has a solution if and only if vector b is a linear combination of the column vectors of A.

Dot product of two vectors

- What is the relation between A, x and the first entry in Ax?

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

- We observe that the first row of A is $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and the the first entry of Ax is $2x_1 + 3x_2 + 4x_3$, which is the sum of the product of corresponding entries in the row and vector x.
- This kind of product is called the dot product of two vectors. Formally, the dot product of vectors $u = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ is

$$u \cdot v = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- So if the product Ax is defined, then the i-th entry in Ax is dot product of the i-th row vector of A and the vector x.

Properties of matrix-vector product Ax

- Theorem 5: If A is an $m \times n$ matrix, and u and v are vectors in $\mathbf{R}^n$, and $c \in \mathbf{R}$ is a scalar, then

  1. $A(u+v) = Au + Av$
  2. $A(cu) = c(Au)$

- Proof. Let the column vectors of A be $a_1, a_2, \dots, a_n$. that is,

  $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$. And let $u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$.

- For part (1):

$$A(u+v) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \dots \\ u_n+v_n \end{bmatrix}$$
$$= (u_1+v_1)a_1 + (u_2+v_2)a_2 + \dots + (u_n+v_n)a_n$$
$$= (u_1a_1 + u_2a_2 + \dots + u_na_n) + (v_1a_1 + v_2a_2 + \dots + v_na_n)$$
$$= Au + Av.$$

- For part (2):

$$A(cu) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ \dots \\ cu_n \end{bmatrix}$$
$$= (cu_1)a_1 + (cu_2)a_2 + \dots + (cu_n)a_n$$
$$= c(u_1a_1 + u_2a_2 + \dots + u_na_n)$$
$$= c(Au).$$

Homogeneous linear systems

- A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbf{R}^m$.
- Such a system $Ax = 0$ always has at least one solution, namely, $x = 0$ (the zero vector in $\mathbf{R}^m$). This zero solution is usually called the trivial solution.
- The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

Examples

- Ex 1: Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- Solution: Let A be the matrix of coefficients of the system and row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since x_3 is a free variable, $Ax = 0$ has nontrivial solutions.

Examples—Cont.

- Continue the row reduction to get the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Solve for basic variables x_1 and x_2 (in terms of free variable x_3), we have $x_1 = 4/3x_3, x_2 = 0$, and x_3 is free.
- As a vector, the general solution of $Ax = 0$ has the following form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_3/3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 v$$

- That is, every solution of $Ax = 0$ in this case is a scalar multiple of vector v.

Parametrix Vector Form

- The equation of the form $x = su + tv$ (with vectors x, u, v and scalars s, t) is called a parametric vector equation of the plane.
- In Example 1, the equation $x = x_3 v$ (with x_3 free), or $x = tv$ (with $t \in \mathbf{R}$), is a parametric vector equation of a line.
- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in parametric vector form.

Solutions of non homogeneous systems

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

- Example 2: Describe all solutions of $Ax = b$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

- Solution: we row reduce the augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- So $x_1 = -1 + 4/3x_3, x_2 = 2$ and x_3 is free.
- As a vector, we can write the solution as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 4/3x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

- So the equation has solution with parametric form $x = p + tv$ with vectors p and v and scalar $t = x_3$.
- Note that the solution set of $Ax = 0$ has the parametric vector equation $x = tv$ with $t \in \mathbf{R}$.
- The solution of $Ax = b$ are obtained by adding a vector p to the solution of $Ax = 0$.
- The vector p itself is just one particular solution of $Ax = b$ (when $t = 0$).

Steps writing a solution set in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution x as a vector whose entries depend on the free variables, if any.
4. Decompose x into a linear combination of vectors (with numeric entries) using the free variables as parameters.

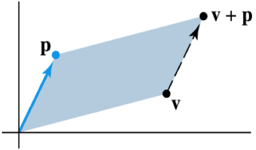
Example

Ex: Express the solution set with the following augmented matrix in parametric vector form

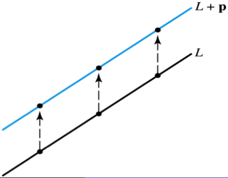
$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -2 & 5 & 1 \\ 0 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solutions to $Ax = 0$ and $Ax = b$

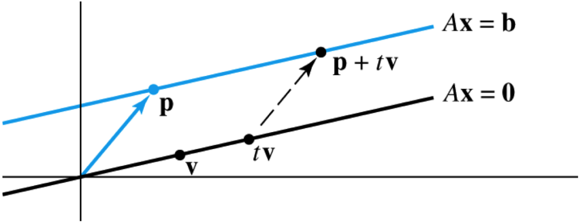
- Now, to describe the solution of $Ax = b$ geometrically, we can think of vector addition as a translation.
- Given vectors p and v in $\mathbf{R}^2$ or $\mathbf{R}^3$, the effect of adding v to p is to move p in a direction parallel to the line through v and 0.
- We say that p is translated by v to $p + v$. See the figure



- If each point on a line L in $\mathbf{R}^2$ or $\mathbf{R}^3$ is translated by a vector p, the result is a line parallel to L.



- Suppose L is the line through 0 and v, described by equation $x = tv$.
- Adding p to each point on L produces the translated line described by equation $x = p + tv$.
- We call $x = p + tv$ the equation of the line through p parallel to v.
- Thus the solution set of $Ax = b$ is a line through p parallel to the solution set of $Ax = 0$. See below figure



- The relation between the solution sets of $Ax = b$ and $Ax = 0$ shown in the figure above generalizes to any consistent equation $Ax = b$, although the solution set will be larger than a line when there are several free variables.

- Theorem 6: Suppose the equation $Ax = b$ is consistent for some given vector b, and let (vector) p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is the general solution of the homogeneous equation $Ax = 0$.

- This theorem says that if $Ax = b$ has a solution, then the solution set is obtained by translating the solution set of $Ax = 0$, using any particular solution p of $Ax = b$ for the translation.