# Section 1.8–1.9 Introduction to Linear Transformation

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# Transformationa

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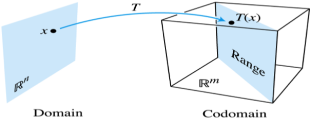
- The set of all images $T(x)$ is called the range of $T$. See the figure on the next slide.




Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

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- The domain of $T$ is $\mathbf{R}^n$, when $A$ has $n$ columns and the codomain of T is $\mathbf{R}^m$, when each column of $A$ has $m$ entries.
- The range of $T$ is the set of all linear combinations of the columns of $A$, because each image $T(x)$ is of the form $Ax$.

# Example

- Ex 1: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(x) = Ax$, so that

$$T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

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- (a). Compute $T(u)$:

$$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

- (b) Solve $T(x) = c$ for $x$. That is, solve $Ax = c$:

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- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

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- Hence $x_1 = 1.5, x_2 = -0.5$ and $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$

- The image of this vector $x$ under $T$ is the given vector $c$.

- (c) Any $x$ whose image under $T$ is $c$ must satisfy the matrix equation in (b). But it is clear from (b) that the matrix equation has a unique solution. So there is exactly one $x$ whose image is $c$.

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- To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

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- So the system is inconsistent. That is, $c$ is not in the range of $T$.

# Shear Transformation

- Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = Ax$ is called a shear transformation.

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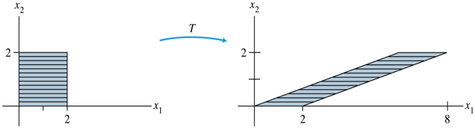

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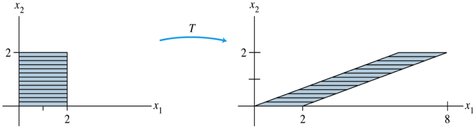


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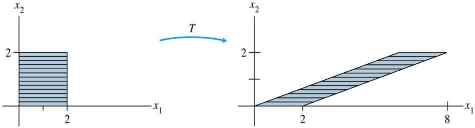


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- $T(\begin{bmatrix} 0 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, and $T(\begin{bmatrix} 2 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

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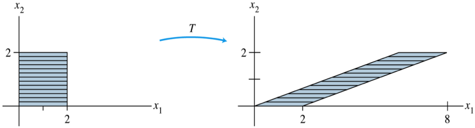


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- $T$ deforms the square as if the top of the square were pushed to the right while the base is held fixed.

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- Note that property (d) implies property (c).
- Any transformation is linear if and only if it satisfies (d).

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- $T$ is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$.

# Superposition principle

- Repeated application of (d) produces a useful generalization:

$$T(c_1 v_1 + c_2 v_2 + \ldots + c_p v_p) = c_1 T(v_1) + c_2 T(v_2) + \ldots + c_p T(v_p)$$

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- Think of $v_1, \ldots, v_p$ as signals that go into a system and $T(v_1), \ldots, T(v_p)$ as the responses of that system to the signals.

# Superposition principle

- Repeated application of (d) produces a useful generalization:

$$T(c_1 v_1 + c_2 v_2 + \ldots + c_p v_p) = c_1 T(v_1) + c_2 T(v_2) + \ldots + c_p T(v_p)$$

- In engineering and physics, the above equation is referred to as a superposition principle.

- Think of $v_1, \ldots, v_p$ as signals that go into a system and $T(v_1), \ldots, T(v_p)$ as the responses of that system to the signals.

- The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the systems response is the same linear combination of the responses to the individual signals.

# The matrix of linear transformation

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- In fact, let $e_i = \begin{bmatrix} 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$ with 1 being in the i-th entry, then $A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$

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&= x_1 T(e_1) + x_2 T(e_2) + \ldots + x_n T(e_n) \\
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- The matrix $A$ is called the standard matrix for $T$.

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So $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

# Onto and one-to-one mappings

- Defn: A mapping $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is said to be onto $\mathbf{R}^m$ if each $b$ in $\mathbf{R}^m$ is the image of at least one $x$ in $\mathbf{R}^n$.

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- Equivalently, $T$ is one-to-one, if for each $b$ in $\mathbf{R}^m$, $T(x) = b$ has at most one solution.

# Example

Ex. Let $T$ be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

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- We also note that $A$ has a free variable $x_3$, for each $b$ in $\mathbf{R}^m$, $Ax = b$ has infinite many solutions. So $T$ is not one-to-one.