Section 2.1 Matrix operations

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$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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- The diagonal entries in an $m \times n$ matrix $A = (a_{ij})$ are $a_{11}, a_{22}, \dots,$, and they are form the main diagonal of A.

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- The $n \times n$ diagonal matrix whose entries are ones is the identity matrix, denoted by I_n.
- An $m \times n$ matrix whose entries are all zero is a zero matrix and is written as 0.
- The two matrices are equal if they have the same size (i.e., the same number of rows and the same number of columns) and if their corresponding entries are equal.

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- Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.
- So if $A = (a_{ij})$ and $B = (b_{ij})$, and $r \in \mathbf{R}$, then

$$A + B = (a_{ij} + b_{ij}), rA = (ra_{ij}), A^T = (a_{ji})$$

Example

Ex1: Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$. Find $A + B, 3A, B^T$.

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- Let A and B be matrices with the same size. Then
  1. $(A^T)^T = A$
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  3. For any scalar r, $(rA)^T = rA^T$

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- If A is $m \times n$, B is $n \times p$ with columns $b_1, b_2, \ldots, b_p$, and x is in $\mathbf{R}^p$, then

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- So when multiply the vector Bx by A, we have

$$A(Bx) = A(x_1 b_1) + A(x_2 b_2) + \ldots + A(x_p b_p) = x_1 A b_1 + x_2 A b_2 + \ldots + x_p A b_p$$

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Soln: note that the (i, j)-entry is the dot product of i-th row of A and j-th column of B. So

$$AB = \begin{bmatrix} 2 \cdot 4 + 3 \cdot 1 & 2 \cdot 3 + 3 \cdot (-2) & 2 \cdot 9 + 3 \cdot 3 \\ 1 \cdot 4 + (-5) \cdot 1 & 1 \cdot 3 + (-5) \cdot (-2) & 1 \cdot 9 + (-5) \cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

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- In terms of transpose, we have $(AB)^T = B^T A^T$.

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- If A is an $n \times n$ matrix, and k is a positive integer, then A^k denote the product of k copies of A: $A^k = AA \dots A$. Moreover, we denote A^0 to eve the identity matrix.