Section 4.2-4.3 Null space, column space, and their bases

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- In other words, $Nul\ A = \{x \in \mathbf{R}^n : Ax = 0\}$.
- One may think $Nul\ A$ to be the set of vectors $x \in \mathbf{R}^n$ that are mapped into the zero vector of $\mathbf{R}^m$ via the linear transformation $x \to Ax$.

Thm: The null space of an $m \times n$ matrix is a subspace of $\mathbf{R}^n$. (Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.)

Pf: $x \in \text{Nul}A \Rightarrow x \in \mathbb{R}^n \Rightarrow \text{Nul}A \subseteq \mathbb{R}^n$

i) $0 \overset{?}{\in} \text{Nul}A$: $A \cdot 0 = 0$

ii) $u, v \in \text{Nul}A \overset{?}{\Rightarrow} u+v \in \text{Nul}A$

$\Uparrow$

$u, v \in \text{Nul}A \Rightarrow Au = 0, Av = 0$

$\Rightarrow A(u+v) = Au + Av = 0 + 0 = 0$

iii) $u \in \text{Nul}A, a \in \mathbb{R} \Rightarrow Au = 0 \Rightarrow A(au) = aAu = a \cdot 0 = 0$

$\Rightarrow au \in \text{Nul}A$

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- Solving the equation $Ax = 0$ amounts to producing an explicit description of $Nul\ A$.

Example

Ex. Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} 3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$R1 \leftrightarrow R2$

$$\xrightarrow{R1+R2} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & -3 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \end{array}$$

$$\begin{cases} x_1 = 0 \\ x_2 = -\frac{1}{2}x_4 + \frac{3}{2}x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}x_4 + \frac{3}{2}x_5 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ \frac{3}{2} \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Null } A = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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- So we row reduce the augmented matrix to reduce echelon form:

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- So x_1 and x_3 are basic variables, and x_2, x_4, x_5 are free variables. And we have $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = 2x_4 + 2x_5$.

- Now we write the solution x in the vector form:

$$x = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ 2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2 u + x_4 v + x_5 w$$

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- Two Remarks:
  1. The spanning set produced by the method in the above Example is automatically linearly independent because the free variables are the weights on the spanning vectors.
  2. When $Nul\ A$ contains nonzero vectors, the number of vectors in the spanning set for $Nul\ A$ equals the number of free variables in the equation $Ax = 0$.

Column space of a matrix

- The column space of an $m \times n$ matrix A, written as $Col\ A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, then $Col\ A = span\{a_1, a_2, \dots, a_n\}$.

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- The notation Ax for vectors in $Col\ A$ also shows that $Col\ A$ is the range of the linear transformation from x to Ax.
- The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $Ax = b$ has a solution for each $b \in \mathbf{R}^m$. .

Examples

Ex: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

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1. Determine if u is in $Nul\ A$. Could u be in $Col\ A$?

$Au = \begin{bmatrix} 0 \\ \times \\ \\ \end{bmatrix}$ $u \notin Nul\ A$

$Col\ A \subseteq \mathbb{R}^3$

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$\underline{Ax = v}$

$Nul\ A \subseteq \mathbb{R}^4$

$v \notin \mathbb{R}^4$

$\underline{v \notin Nul\ A}$

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Sol: We compute Au and find that it is not zero, which means that u is not a solution to $Ax = 0$. So u is not in $Nul\ A$.

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- To see if v is in *Col* A, we row reduce $\begin{bmatrix} A & v \end{bmatrix}$ to an echelon form:

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- The equation $Ax = v$ is consistent. So v is in $Col\ A$.
- As $v \in \mathbf{R}^3$, v cannot be in $Nul\ A$, which is a subspace of $\mathbf{R}^4$.

- Let T be a linear transformation from V to W.

$$\begin{cases} T(u+v) = T(u) + T(v) \\ T(cu) = cT(u) \end{cases}$$

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- The range of T is the set of all vectors in W of the form $T(x)$ for some x in V.
- The kernel of T is a subspace of V, and the range of T is a subspace of W.

Ex. Let V the vector space of all real-valued functions f on $[a, b]$ which are differentiable and whose derivatives are continuous functions on $[a, b]$, and let W be the vector space of continuous functions on $[a, b]$. Let $D : V \to W$ be the transformation so that $D(f) = f'$.

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- One can show D is a linear transformation.

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$$

$$D(cf) = (cf)' = c \cdot f' = c \cdot D(f)$$

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- One can show D is a linear transformation.
- What is the kernel and range of D?

$\text{Ker } D = \{f \in V : D(f) = 0\} = \{f \in V : f' = 0\}$

$= \{f = \text{Constant}\}$

$\text{ran } D = \{D(f) : f \in V\} = V$

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Ex. Let V the vector space of all real-valued functions f on $[a, b]$ which are differentiable and whose derivatives are continuous functions on $[a, b]$, and let W be the vector space of continuous functions on $[a, b]$. Let $D : V \to W$ be the transformation so that $D(f) = f'$.

- One can show D is a linear transformation.
- What is the kernel and range of D?
- The kernal is the set of constant functions on $[a, b]$, and the range of D is the set W.

Linear independent sets

- An indexed set of vectors $\{v_1, \ldots, v_p\}$ in V is said to be linearly independent if the vector equation $c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$ has only the trivial solution $c_1 = c_2 = \ldots = c_p = 0$.

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- Theorem 4: An indexed set $\{v_1, , v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, v_2, \ldots, v_{j-1}$.

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- Definition: Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1, \ldots, b_p\}$ in V is a basis for H if

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- When $H \neq V$, condition (2) includes the requirement that each of the vectors $b_1, \ldots, b_p$ must belong to H, because $Span\{b_1, \ldots, b_p\}$ contains $b_1, \ldots, b_p$.

Examples

Ex1: Let $A = [a_1\ a_2\ \ldots\ a_n]$ be an invertible matrix. Then the columns of A form a basis for $\mathbf{R}^n$, because they are linear independent and they span $\mathbf{R}^n$ (by the Invertible Matrix Theorem).

$\forall b$

$Ax = b \Rightarrow x = A^{-1}b$

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Ex2: Let $e_1, e_2, \ldots, e_n$ be the columns of the $n \times n$ matrix, I_n. That is, $e_1 = [1\ 0\ \ldots\ 0]^T, e_2 = [0\ 1\ \ldots\ 0]^T, \ldots, e_n = [0\ 0\ \ldots\ 1]^T$. The set $\{e_1, e_2, \ldots, e_n\}$ is called the standard basis for $\mathbf{R}^n$.

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Suppose that $c_0, c_1, \ldots, c_n$ satisfy $c_0 \cdot 1 + c_1 t + c_2 t^2 + \ldots + c_n t^n = 0$. The only way for a polynomial to be the zero polynomial is that the coefficients are all zeros. So $c_0 = c_1 = \ldots = c_n = 0$.

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This means that S is an independent set.

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2. If $H \neq \{0\}$, some subset of S is a basis for H.

Basis for *Nul A* and *Col A*

- From previous example, we have already known how to find basis for *Nul A*: we find the free variables, and write all variables in terms of the free variables, then find the vector from of the general solution for $Ax = 0$, and we can read the basis from the vector from.

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  Row operations can change the column space of a matrix. The columns of an echelon form B of A are often not in the column space of A.

Example

Ex. Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$. Find a basis for *Col* A.

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Sol: We may first row reduce A:

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- Therefore a basis for *Col* A is the following $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix}$.

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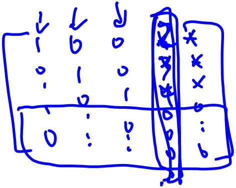

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- This leaves the pivot columns of A as a basis for *Col* A.

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- A basis is also a linearly independent set that is as large as possible.
- If S is a basis for V, and if S is enlarged by one vector—-say, w—-from V, then the new set cannot be linearly independent, because S spans V, and w is therefore a linear combination of the elements in S.