Section 4.4 Coordinate systems and 4.7 Change of basis

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Unique Representation Theorem

Thm (Unique Representation Theorem) let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V. Then for each $x \in V$, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that $x = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$.

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- Then we have $0 = x - x = (c_1 - d_1)b_1 + (c_2 - d_2)b_2 + \ldots + (c_n - d_n)b_n$.
- Since $\mathcal{B}$ is linearly independent, all the weights must be zeros. That is, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$.

Coordinates

- Definition: Suppose $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for V and x is in V. The coordinates of x relative to the basis B (or the B-coordinate of x) are the weights $c_1, c_2, \ldots, c_n$ so that

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- The mapping $x \to [x]_B$ is the coordinate mapping determined by B.
- When a basis B is fixed for $\mathbf{R}^n$, it is easy to find the B-coordinate vector of a specified x.

Example

Ex. Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $B = \{b_1, b_2\}$. Find the coordinate vector $[x]_B$ of x.

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Coordinates in $\mathbf{R}^n$

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- Then the vector equation $x = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$ can be written as

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- This is not only true from $\mathbf{R}^n$ to $\mathbf{R}^n$, but also holds for a vector space with a basis of n vectors to $\mathbf{R}^n$.

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Thm. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \to [x]_B$ is a one-to-one and onto linear transformation from V to $\mathbf{R}^n$.

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- That is, $[u+v]_B = [u]_B + [v]_B$ and $[cu]_B = c[u]_B$.
- Let $u = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$ and $v = d_1 b_1 + d_2 b_2 + \ldots + d_n b_n$. Then

$$u + v = (c_1 + d_1)b_1 + (c_2 + d_2)b_2 + \ldots + (c_n + d_n)b_n$$

and

$$cu = (cc_1)b_1 + (cc_2)b_2 + \ldots + (cc_n)b_n$$

- So $[u+v]_B = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \dots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{bmatrix} = [u]_B + [v]_B.$

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- We skip the proof for one-to-one and onto (homework).

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- The coordinate mapping in Theorem 8 is an important example of an isomorphism from V onto W.
- In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism from V onto W.
- The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces, if there is an isomorphism between them.

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- Writing the vector as the columns of a matrix A, and determine their independence by row reducing the augmented matrix for $Ax = 0$:

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Furthermore, $c_3 = 2c_2 - 5c_1$. So we have

$$3+2t = 2(4+t+5t^2) - 5(1+2t^2).$$

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- In applications, we may know the B-coordinate for the vector, but need to know its C-coordinate for another basis C.
- We will give a way to build connection between $[x]_B$ and $[x]_C$, in particular, we will find the change-of-coordinates matrix from B to C.

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Sol. Note that $[x]_C = [3b_1 + b_2]_C = 3[b_1]_C + [b_2]_C = \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

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As $[b_1]_C = [4 \;\, 1]^T, [b_2]_C = [-6 \;\, 1]^T$, we have

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

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$$[x]_C = P_{B \to C} \cdot [x]_B.$$

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- Note that the columns of $P_{B \to C}$ are linearly independent and it is a square matrix, it must be an invertible matrix. So we also have the following

$$[x]_B = P_{B \to C}^{-1} \cdot [x]_C.$$

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- Note that the columns of $P_{B \to C}$ are linearly independent and it is a square matrix, it must be an invertible matrix. So we also have the following

$$[x]_B = P_{B \to C}^{-1} \cdot [x]_C.$$

- In other words,

$$P_{C \to B} = P_{B \to C}^{-1}$$

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- To find $P_{B\to C}$, we just need to find $P_{C\to B}^{-1}$:

$$P_{B\to C} = P_{C\to B}^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$