Section 4.5-~~4.6~~ Dimension ~~and rank of~~ vector spaces

& 4.6 rank of a matrix

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Dimension of a vector space

- In the last section, we show that a vector space V with a basis B containing n vectors is isomorphic to $\mathbf{R}^n$.

$$v \longrightarrow [v]_B \in \mathbb{R}^n$$

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Thm9 If a vector space V has a basis $B = \{b_1, b_2, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

(i.e., a basis cannot have more than n vectors)

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- So there exist scalars $c_1, c_2, \ldots, c_p$, not all zero, such that

$$c_1[u_1]_B + c_2[u_2]_B + \ldots + c_p[u_p]_B = 0$$

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- Since c_i are not all zero, $\{u_1, \ldots, u_p\}$ is linearly dependent.

- Theorem 10: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

B_1 — n vectors

B_2 — m vectors

$$\left.\begin{array}{l} m \le n \\ n \le m \end{array}\right\} \Rightarrow m = n$$

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- Thus B_2 consists of exactly n vectors.

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- 0-dimension: zero subspace
- 1-dimension: line passing through the origin
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- 3-dimension: the $\mathbf{R}^3$.

Example

Ex. Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d + 1 \end{bmatrix} : a, b, c, d \in \mathbf{R} \right\}.$$

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So $H = Span\{v_1, v_2, v_3, v_4\}$.

$A = [v_1\ v_2\ v_3\ v_4]$

$= Col(A)$

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So $dim H = 3$.

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- Otherwise, there is some v_{k+1} in H that is not in $Span\ S$. But then $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ is linearly independent, because no vector in the set can be a linear combination of vectors that precede it.

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- $dim\ H \leq dim\ V$ follows as a corollary.

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- Now suppose S has p elements and span V. Then by the Spanning Set Theorem, S contains a basis. But a basis contains p elements, so S must be a basis.

Dimension of *Nul A* and *Col A*

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To see the dimension of *Nul A*, we suppose that $Ax = 0$ has k free variables. Then each solution to $Ax = 0$ can be expression a linear combination of k independent vectors, one for each free variable. So the k vectors form a basis for *Nul A*.

Example

Ex. Find the dimension of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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Sol. Row reduce the augmented matrix $[A\ 0]$ to echelon form:

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$\dim(\text{Col}(A)) = 2$

$\dim(\text{Nul}(A)) = 3$

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There are two pivot columns, so $dim\ Col\ A = 2$.

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- It is indeed true....

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- One way to study $Row\ A$ is to study $Col\ A^T$. But there are more directed ways to do it!

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- If B is in echelon form, then the nonzero rows are linearly independent, because no nonzero row is a linearly combinations of the nonzero row below it.

- Theorem: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Pf. If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A.

It follows that any linear combinations of rows of B are automatically linear combinations of rows of A.

Thus the row space of B is contained in the row space of A.

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  Thus the nonzero rows of B form a basis of the row space of B and A.

Example

Ex. Find bases for the row space, column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

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- By the theorem, the basis for row space of A is the first three rows of B: $\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$.

- The pivot columns of B (thus A) are the the first, second and fourth columns. So a basis for column space of A is the first, second, and fourth columns of A: $\{(-2,-5,8,0,-17)^T,(1,3,-5,1,5)^T,(1,7,-13,5,-3)^T\}$.

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- Therefore a basis for *Nul A* is $\{(-1,2,1,0,0)^T,(-1,-3,0,5,1)^T\}$.

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- The second part follows from $rank\ A = \dim Row\ A = \dim Col\ A$ and $rank\ A + \dim Nul\ A = n$.

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Sol. (a) the rank of A is $9 - 2 = 7$.

(b) A 6×9 matrix cannot have a two-dimensional null space, for otherwise, the rank of A is $9 - 2 = 7$, which equals the dimension of column space, but the column space is a subspace of $\mathbf{R}^6$.

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