Section 5.1-2 Eigenvalues and eigenvectors

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Eigenvectors and Eigenvalues

- Definition: An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

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- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Example

Ex. Show that 7 is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

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$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

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x_2 ↓

$x_1 = x_2$

x_2 free

- So it has nontrivial solutions, and the general solution has the form $x_2[1\ 1]^T$.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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eigenspace $= \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

- So 7 is eigenvalue and the corresponding eigenvectors have the form $x_2[1\ 1]^T$ with $x_2 \neq 0$.

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Ex. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

$(A - 2I)x = 0$

$\text{Nul}(A - 2I)$

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$x_1 = \frac{1}{2}x_2 - 3x_3$

- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

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- So a basis for the eigenspace for 2 is $\{[1/2\ 1\ 0]^T, [-3\ 0\ 1]^T\}$.

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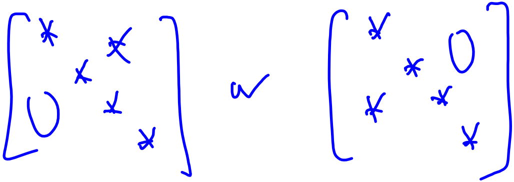
THM. The eigenvalues of a triangular matrix are the entries on its main diagonal.



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- It is clear that $(A - \lambda I)x = 0$ has a free variable if and only if at least one entry on the diagonal of $A - \lambda I$ is zero.
- This happens if and only if λ equals one of the a_{11}, a_{22}, a_{33} in A.

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Pf: Suppose $\{v_1, v_2, \ldots, v_r\}$ is linearly dependent.

- Since v_1 is nonzero, one of the vectors in the set is a linear combination of the preceding vectors.
- Let p be the least index such that v_{p+1} is a linear combination of the preceding (linearly independent) vectors.

- Then there exist scalars $c_1, c_2, \ldots, c_p$ so that

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$$c_1 = c_2 = \ldots = c_p = 0$$

- It follows that $v_{p+1} = 0$, a contradiction.

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- To show the full statement, assume that $\det(A - \lambda I) = 0$. Then $A - \lambda I$ is not invertible. It follows that $(A - \lambda I)x = 0$ has nontrivial solutions. That is, $Ax = \lambda x$ has nontrivial solutions, and λ is an eigenvalue of A.

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$\lambda \quad -3$
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- So we have

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = \lambda^2 + 4\lambda - 21 = 0$$

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$(\lambda + 3)(\lambda + 7) = 0$

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- Solve the quadratic equation, we have $\lambda = 3$ or -7.
- So the eigenvalues of A are 3 and -7.

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Sol. The characteristic equation is $\det(A - \lambda I) = 0$, which is

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

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$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

- The characteristic equation is $(\lambda - 5)^2(\lambda - 3)(\lambda - 1) = 0$.
- In terms of polynomial, it is

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

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$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2)$$

- So the eigenvalues are 0 (multiplicity 4), 6 (multiplicity 1) and -2 (multiplicity 1).

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- Change A into $P^{-1}AP$ is called a similarity transformation.

Similar matrices

THM. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

$$\det(A-\lambda I) = \det(P^{-1}BP-\lambda I)$$

$$= \det(P^{-1}BP - P^{-1}\lambda P)$$

$$= \det(P^{-1}(B-\lambda I)P)$$

$$= \det(P^{-1}) \cdot \det(B-\lambda I) \cdot \det(P)$$

$$\det(B-\lambda I) = \det(P^{-1}P) \cdot \det(B-\lambda I)$$

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- That is, B and A has the same characteristic equation, so same eigenvalues with the same multiplicities.