Section 6.1 Inner Product, Length, and Orthogonality

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Sol. $u \cdot v = v \cdot u = (2)(3) + (-5)(2) + (-1)(-3) = -1.$

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- $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$.

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$$

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- $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$.

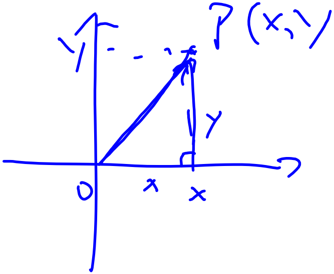
- A more general property is true:

$$(c_1 u_1 + c_2 u_2 + \ldots + c_p u_p) \cdot w = c_1(u_1 \cdot w) + c_2(u_2 \cdot w) + \ldots + c_p(u_p \cdot w)$$

Length of a vector

- Recall that for a point $P(x, y)$, the length of OP is $\sqrt{x^2 + y^2}$. And if we let u be the vector P corresponds to, then the length of OP is $\sqrt{u \cdot u}$.

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$



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- So $||v||^2 = v \cdot v$.
- Let c be a scalar. Then $||cv|| = |c| \ ||v||$.

Pf: $||cv|| = \sqrt{(cv)\cdot(cv)} = \sqrt{c^2(v \cdot v)} = |c| \cdot \sqrt{v \cdot v}$

$= |c| \cdot ||v||$

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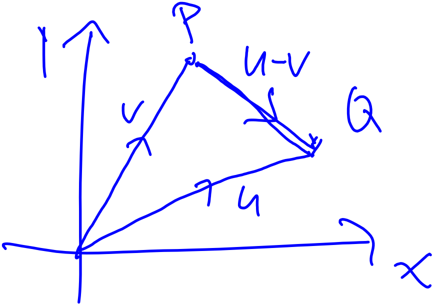
- Then normalize v and get a unit vector: $\frac{v}{||v||} = [1/3 \ \ -2/3 \ 2/3 \ 0]$.

Distance in $\mathbf{R}^n$

- For vectors $u, v \in \mathbf{R}^n$, the distance between u and v, written as $dist(u, v)$, is the length of the vector $u - v$. That is, $dist(u, v) = ||u - v||$.



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- So the distance is $dist(u, v) = ||u - v|| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$.

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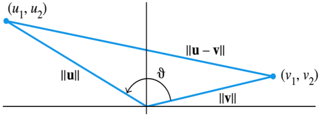
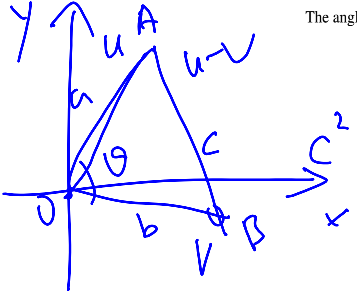


The angle between two vectors.



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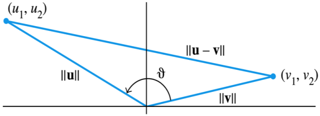

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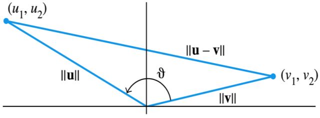

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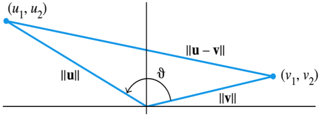

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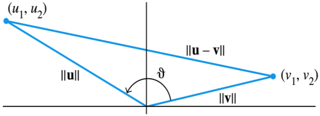

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- We define the angle between two vectors using the above formula.

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$0 = \cos 90^\circ = \cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}$

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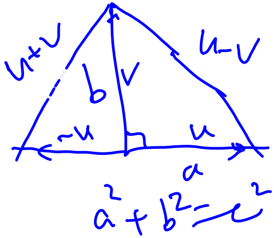

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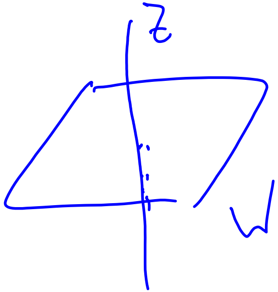
- So $dist(u, v) = dist(u, v)$ if and only if $uv = 0$.

Orthogonal complements

- If a vector z is orthogonal to every vector in a subspace W of $\mathbf{R}^n$, then z is said to be orthogonal to W.

$z \perp W$



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- If a vector z is orthogonal to every vector in a subspace W of $\mathbf{R}^n$, then z is said to be orthogonal to W.
- The set of all vectors z that are orthogonal to W is called the orthogonal complement of W and is denoted by $W^\perp$ (read as "W perp").

$$W^\perp = \{ z : z \perp W \}$$

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THM A vector x is in $W^\perp$ if and only if x is orthogonal to every vector in a set that spans W.

Orthogonal complements

$W^\perp = \{x : x \perp W\}$

$(W^\perp)^\perp = W$

THM $W^\perp$ is a subspace of $\mathbf{R}^n$.

$u \in W^\perp$, $x \in W$ $\quad u \cdot x = 0$

Pf: ① $0 \in W^\perp$: $0 \cdot u = 0$, $\forall u \in W$

② $u, v \in W^\perp \Rightarrow u+v \in W^\perp$:

$\forall x \in W$, $u \cdot x = 0$, $v \cdot x = 0$ $\quad (u+v) \cdot x = u \cdot x + v \cdot x = 0 + 0 = 0$

$u + v \perp x$

③ $u \in W^\perp$, $a \in \mathbb{R} \Rightarrow au \in W^\perp$:

$\forall x \in W$, $u \cdot x = 0$ $\quad (au) \cdot x = a(u \cdot x) = a \cdot 0 = 0$

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THM Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(Row\ A)^{\perp} = Nul\ A \qquad (Col\ A)^{\perp} = Nul\ A^T$$

$$(Col\ (A^T)^T)^{\perp} = Nul\ A^T$$

$B \qquad B$

$$(Col\ B^T)^{\perp} = Nul\ B$$

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Pf. Note that $x \in Nul\ A$ if and only if $Ax = 0$. That is, x is orthogonal to every row vector of A. So we have the conclusions.