Section 6.2-6.3 Orthogonal sets and orthogonal projections

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Orthogonal sets

- A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbf{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is $u_i \cdot u_j = 0$ whenever $i \neq j$.

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- Theorem 4: If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in $\mathbf{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof of Theorem 4

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&= c_1(u_1 \cdot u_1) + \ldots + c_p(u_p \cdot u_1) = c_1(u_1 \cdot u_1)
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- Since u_1 is nonzero, $u_1 \cdot u_1$ is not zero and so $c_1 = 0$.
- Similarly, $c_2, \ldots, c_p$ must be zero.
- Thus S is linearly independent.

Orthogonal basis

- An orthogonal basis for a subspace W of $\mathbf{R}^n$ is a basis for W that is also an orthogonal set.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ basis for } \mathbb{R}^3$$

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- Theorem 5: Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbf{R}^n$. For each y in W, the weights in the linear combination $y = c_1 u_1 + \ldots + c_p u_p$ are given by $c_j = \frac{y \cdot u_j}{u_j \cdot u_j}$ for $j = 1, \ldots, p$.

Remark: easy to find $[y]_U$.

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- To find c_j, we can similarly compute $y \cdot u_j$ and solve for c_j.

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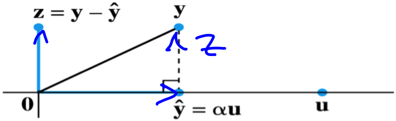
- We wish to write $y = \hat{y} + z$, where $\hat{y} = \alpha u$ for some scalar α and z is some vector orthogonal to u.



Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

$z \perp u$; $\quad (y - \hat{y}) \cdot u = 0 \quad y \cdot u - \hat{y} \cdot u = 0$

$y \cdot u - \alpha u \cdot u = 0 \quad \alpha u \cdot u = y \cdot u \Rightarrow \alpha = \frac{y \cdot u}{u \cdot u}$

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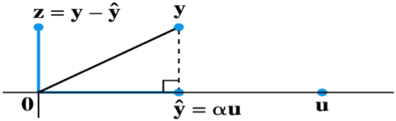

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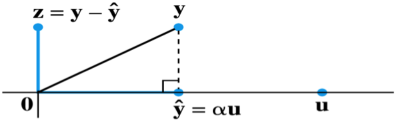

Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

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- That is, $y = \hat{y} + z$ with z orthogonal to u if and only if $\alpha = \frac{y \cdot u}{u \cdot u}$ and $\hat{y} = \frac{y \cdot u}{u \cdot u} u$.

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- That is,

$$\hat{y} = proj_L y = \frac{y \cdot u}{u \cdot u} u$$

Example

Ex. Let $y = [7\ 6]^T$ and $u = [4\ 2]^T$. Find the orthogonal projection of y onto u. Then write y as the sum of two orthogonal vectors, one in $Span\{u\}$ and one orthogonal to u.

$$\text{Proj}_L y = \frac{y \cdot u}{u \cdot u} u = \frac{4\cdot 7 + 6\cdot 2}{4\cdot 4 + 2\cdot 2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$z = y - \text{Proj}_L y = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

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- And the component of y orthogonal to u is $y - \hat{y} = [7\ 6]^T - [8\ 4]^T = [-1\ 2]^T$.
- That is, $[7\ 6] = [8\ 4] + [-1\ 2]$.

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- Any nonempty subset of $\{e_1, \cdots, e_n\}$ is orthonormal, too

Orthonormal columns of a matrix

- Theorem 6: an $n \times m$ matrix U has orthonormal columns if and only if $U^T U = I$.

To check $u_1, u_2, \ldots, u_m$ to be orthonormal,

$$U = [u_1\ u_2\ \cdots\ u_n]$$

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- From above conditions, the theorem follows immediately.

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- The above properties say that the linear mapping $x \to Ux$ preserves lengths and orthogonality.

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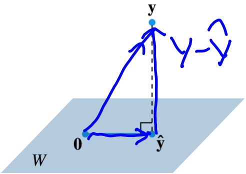
- Given a vector y and a subspace W in $\mathbf{R}^n$, there is a vector $\hat{y}$ in W such that (1) $\hat{y}$ is the unique vector in W for which $y - \hat{y}$ is orthogonal to W, and (2) $\hat{y}$ is the unique vector in W closest to y.



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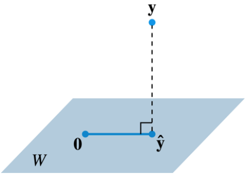

- These two properties of $\hat{y}$ provide the key to finding the least-squares solutions of linear systems.

- Theorem 8: Let W be a subspace of $\mathbf{R}^n$. Then each y in $\mathbf{R}^n$ can be written uniquely in the form $y = \hat{y} + z$, where $\hat{y}$ is in W and z is in $W^{\perp}$.

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$z \perp u_i$ $\quad (y - \hat{y}) \cdot u_i = 0$

$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$

$y \cdot u_i - \hat{y} \cdot u_i = 0$

$y \cdot u_i = c_i u_i \cdot u_i$

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- To show the the decomposition is unique, suppose y can also be written as $y = \hat{y}_1 + z_1$, with $\hat{y}_1$ in W and z_1 in $W^{\perp}$.

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- Hence $v \cdot v = 0$, and implies that $v = 0$. So $\hat{y} = \hat{y}_1$ and $z = z_1$.

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Ex. Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{u_1, u_2\}$ is an orthogonal basis for $W = Span\{u_1, u_2\}$. Write y as the sum of a vector in W and a vector orthogonal to W.

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

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- In this case, $proj_W y = y$. In particular, if y is in $W = Span\{u_1, \ldots, u_p\}$, then $proj_W y = y$.

The best approximation theorem

- Theorem 9. Let W be a subspace of $\mathbf{R}^n$, let y be any vector in $\mathbf{R}^n$, and let $\hat{y}$ be the orthogonal projection of y onto W. Then $\hat{y}$ is the closest point in W to y, in the sense that $||y - \hat{y}|| < ||y - v||$ for all v in W distinct from $\hat{y}$.

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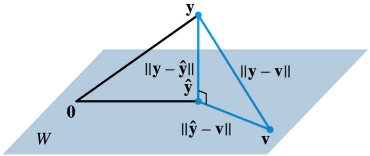
- Inequality $||y - \hat{y}|| < ||y - v||$ leads to a new proof that $\hat{y}$ does not depend on the particular orthogonal basis used to compute it.
- If a different orthogonal basis for W were used to construct an orthogonal projection of y, then this projection would also be the closest point in W to y, namely, $\hat{y}$.

Proof of Theorem 9

PF. Take v in W distinct from $\hat{y}$. Then $\hat{y} - v$ is in W.



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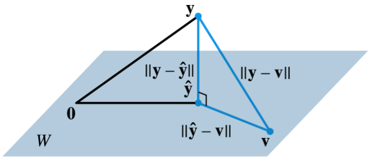

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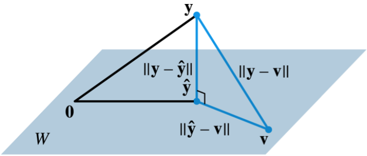

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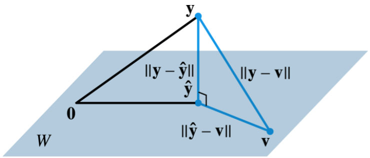

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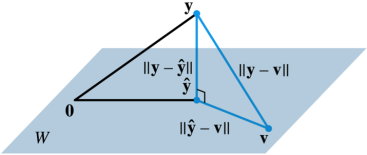

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- Now $||\hat{y} - v||^2 > 0$ because $\hat{y} - v \neq 0$, and so inequality follows immediately.

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- So the distance is $||y - \hat{y}|| = \sqrt{(y - \hat{y}) \cdot (y - \hat{y})} = \sqrt{45}$.

Properties of Orthogonormal projections

- Theorem 10. If $\{u_1, \ldots, u_p\}$ is an orthogonormal basis for a subspace W of $\mathbf{R}^n$, then

$$proj_W y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \ldots + (y \cdot u_p)u_p$$

If $U = [u_1 \; u_2 \; \ldots \; u_p]$, then $proj_W y = UU^T y$ for all $y \in \mathbf{R}^n$.