Section 7.1 Diagonalization of symmetric matrices and 7.2 Quadratic forms

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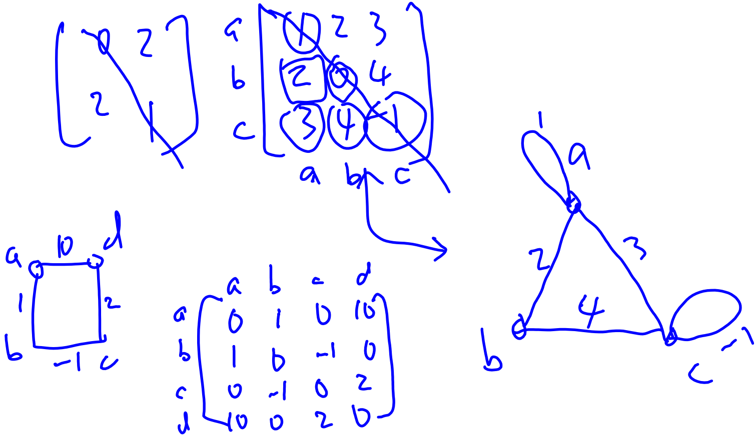
Symmetric matrix

- A symmetric matrix is a matrix A such that $A^T = A$.



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- Its main diagonal entries are arbitrary, but its other entries occur in pairs on opposite sides of the main diagonal.

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$Av_1 = \lambda_1 v_1$

Proof: Let v_1 and v_2 be eigenvectors that correspond to distinct eigenvalues, say λ_1 and λ_2. We show that $v_1 \cdot v_2 = 0$. Note that

$$\begin{aligned}\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 = (v_1^T A^T) v_2 \\ &= v_1^T (A^T v_2) = v_1^T (Av_2) = v_1^T (\lambda_2 v_2) = \lambda_2 (v_1^T v_2) \\ &= \lambda_2 v_1 \cdot v_2.\end{aligned}$$

It follows that $(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$. So $v_1 \cdot v_2 = 0$.

Orthogonally diagonalizable matrix

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- So A should be symmetric.
- Theorem: An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.

Example

- Ex: Orthogonally diagonalize the matrix $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3. \end{bmatrix}$

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- First of all, its charateristic equation is $det(A - \lambda I) = (\lambda - 7)^2(\lambda + 2)$. So its eigenvalues are 7 (with multiplicity 2) and -2.

$$det\left(\begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}\right)$$

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- The bases for the eigenspaces are

$$\lambda = 7 : v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}; \qquad \lambda = -2 : v_3 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$(A - 7I)x = 0$$

$$A - 7I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 x_3

$$x_1 = -\frac{1}{2}x_2 + x_3 \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} =$$

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- The projection of v_2 onto v_1 is $\frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$, so the component of v_2 orthogonal to v_1 is $z_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} -\frac{1}{4} \\ 1 \\ \frac{1}{4} \end{pmatrix}$

Gram-Schmidt process (6.4)

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- So $\{v_1, z_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$.

- Normalize v_1 and z_2 to get a orthonormal basis for the eigenspace for $\lambda = 7$:

$u_1 = \frac{v_1}{\|v_1\|}$

$$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{pmatrix}$$

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- Let

$$P = [u_1 \ u_2 \ u_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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- then P orthogonally diagonalizes A, and $A = PDP^T$.

- The set of eigenvalues of a matrix A is sometimes called the spectrum of A.

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  3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

  4. A is orthogonally diagonalizable.

Spectral decomposition

- Suppose that $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $u_1, \ldots, u_n$ of A and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are in the diagonal matrix D. Then

$$A = PDP^T = [u_1 \;\; u_2 \;\; \ldots \;\; u_n] \begin{bmatrix} \lambda_1 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \begin{pmatrix} u_1^T \\ u_2^T \\ \ldots \\ u_n^T \end{pmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T$$

$n \times n$ matrix

$$u_1 u_1^T = \begin{bmatrix} u_{11} \\ \vdots \\ u_{1n} \end{bmatrix}_{n\times 1} \left[u_{11} \cdots u_{1n} \right]_{1 \times n}$$

$$= (\lambda_1 u_1 \;\; \lambda_2 u_2 \cdots \lambda_n u_n) \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix}$$

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$$
\begin{aligned}
A =& PDP^T = [u_1 \ u_2 \ \ldots \ u_n] \begin{bmatrix} \lambda_1 & 0\ldots & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \begin{pmatrix} u_1^T \\ u_2^T \\ \ldots \\ u_n^T \end{pmatrix} \\
=& \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T
\end{aligned}
$$

- The expression $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T$ is called a spectral decomposition of A, because if breaks A into pieces determined by the spectrum (eigenvalues) of A.

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- Each term in the decomposition is an $n \times n$ matrix of rank 1.

Example

- Ex:

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

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- Then

$$A = 8u_1u_1^T + 3u_2u_2^T = \begin{bmatrix} \frac{32}{5} & \frac{16}{5} \\ \frac{16}{5} & \frac{8}{5} \end{bmatrix} + \begin{bmatrix} \frac{3}{5} & -\frac{6}{5} \\ -\frac{6}{5} & \frac{12}{5} \end{bmatrix}$$

Quadratic forms

- A quadratic form is a function defined on $\mathbf{R}^n$ whose value at a vector x in $\mathbf{R}^n$ can be computed by an expression of the form $Q(x) = x^T Ax$, where A is an $n \times n$ symmetric matrix.

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- The matrix A is called the matrix of the quadratic form.

$$2x_1^2 + 3x_1x_2 - 2x_1x_3 + 4x_2x_3 + x_2^2 + 5x_3^2 = x^T A x$$

$$A = \begin{pmatrix} 2 & 3/2 & -1 \\ 3/2 & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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- Ex: let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Compute $x^T A x$ for the following matrices:

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$$

x^TAx

$= (x_1\ x_2)\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (1×2, 2×2)

$= (4x_1\ \ 3x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_1^2 + 3x_2^2$

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- $x^T Ax = 4x_1^2 + 3x_2^2$.
- $x^T Bx = 3x_1^2 - 4x_1x_2 + 7x_2^2$.
- The term $-4x_1x_2$ is called a cross-product term.

Change of variable in a quadratic form

- If x represents a variable vector in $\mathbf{R}^n$, then a change of variable is an equation of the form $x = Py$, or $y = P^{-1}x$, where P is an invertible matrix and y is a new variable vector in $\mathbf{R}^n$.

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- By make a change of variable on x^TAx, we have

$$x^TAx = (Py)^TA(Py) = y^TP^TAPy = y^T(P^TAP)y$$

we obtain a new matrix of the quadratic form P^TAP.

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- Since A is symmetric, there is an orthogonal matrix P such that $P^T AP$ is a diagonal matrix D, and the quadratic form above becomes $y^T Dy$, which contains no cross-product terms.

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  we obtain a new matrix of the quadratic form $P^T AP$.
- Since A is symmetric, there is an orthogonal matrix P such that $P^T AP$ is a diagonal matrix D, and the quadratic form above becomes $y^T Dy$, which contains no cross-product terms.
- Theorem: Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $x = Py$, that transforms the quadratic form $x^T Ax$ into a quadratic form $y^T Dy$ with no cross-product term.

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- Orthogonally diagonalize A: the eigenvalues of A are 3 and -7, and the associated unit eigenvectors are $\lambda = 3 : [\frac{2}{\sqrt{5}} \;\; -\frac{1}{\sqrt{5}}]^T$ and $\lambda = -7 : [\frac{1}{\sqrt{5}} \;\; \frac{2}{\sqrt{5}}]^T$.

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$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

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- then $A = PDP^T$ and $D = P^TAP$. Let $x = Py$, then

$$x_1^2 - 8x_1x_2 - 5x_2^2 = y^TDy = 3y_1^2 - 7y_2^2$$

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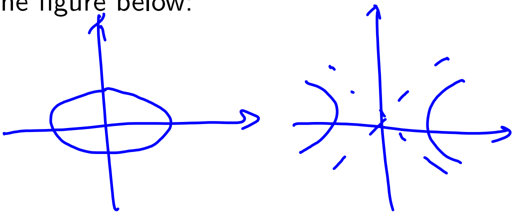

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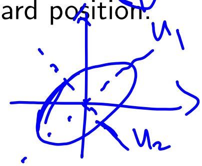

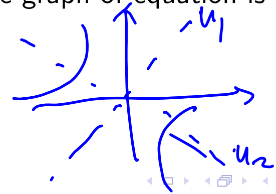

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