CONNECTIVITIES FOR \( k \)-KNITTED GRAPHS AND FOR MINIMAL COUNTEREXAMPLE TO HADWIGER’S CONJECTURE

KEN-ICHI KAWARABAYASHI† AND GEXIN YU‡

Abstract. For a given subset \( S \subseteq V(G) \) of a graph \( G \), the pair \((G, S)\) is knitted if for every partition of \( S \) into non-empty subsets \( S_1, S_2, \ldots, S_t \), there are disjoint connected subgraphs \( C_1, C_2, \ldots, C_t \) in \( G \) so that \( S_i \subseteq C_i \). A graph \( G \) is \( \ell \)-knitted if \((G, S)\) is knitted for all \( S \subseteq V(G) \) with \(|S| = \ell\). In this paper, we prove that every \( 9\ell \)-connected graph is \( \ell \)-knitted.

Hadwiger’s Conjecture states that every \( k \)-chromatic graph contains a \( K_k \)-minor. We use the above result to prove that the connectivity of minimal counterexamples to Hadwiger’s Conjecture is at least \( k/9 \), which was proved to be at least \( 2k/27 \) in [4].

1. Introduction

One of the most interesting problems in graph theory is Hadwiger’s Conjecture, which states that every \( k \)-chromatic graph has a \( K_k \)-minor, where a graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) by contracting edges.

It is known that Hadwiger’s Conjecture holds for \( k \leq 6 \). Wagner [11] in 1937 proved that the case \( k = 5 \) is equivalent to Four Color Theorem. About 60 years later, Robertson, Seymour and Thomas [8] proved that the case \( k = 6 \) is also equivalent to the Four Color Theorem. In their proof, minimal counterexamples, which are also called “contraction-critical non-complete graphs”, play an important role. Kawarabayashi and Toft [5] showed that 7-chromatic graphs contain a \( K_7 \)-minor or a \( K_{4,4} \)-minor, in which the connectivity property of minimal counterexamples are, again, really important.

Many researchers have considered the connectivity property of contraction-critical graphs. Dirac [2] proved that every \( k \)-contraction-critical graph is 5-connected for \( k \geq 5 \), and Mader [7] extended 5-connectivity to the deep result that every \( k \)-contraction-critical graph is 7-connected for \( k \geq 7 \) and every 6-contraction-critical graph is 6-connected. Toft [10] proved that \( k \)-contraction-critical graphs are \( k \)-edge-connected. Kawarabayashi [4] proved the first general result on the vertex connectivity of minimal counterexamples to Hadwigers Conjecture.

Theorem 1 (Kawarabayashi [4]). For all positive integers \( k \), every minimal (with respect to the minor relation) \( k \)-chromatic counterexample to Hadwiger’s Conjecture is \( \lceil \frac{2k}{27} \rceil \)-connected.

In the proof of the above theorem, the main tool used was so-called \( k \)-linked graphs. A graph \( G \) is \( k \)-linked if for every \( 2k \) distinct vertices \( u_1, v_1, u_2, v_2, \ldots, u_k, v_k \) in \( G \), there are \( k \) disjoint paths \( P_1, P_2, \ldots, P_k \) such that \( P_i \) connects \( u_i \) and \( v_i \). \( k \)-linked graphs are very well-studied and play a very important role in the study of graph structures.

In this paper, we improve the result in Theorem 1, by studying a notion called “knitted graph” introduced by Bollobás and Thomassen [1].
For $1 \leq m \leq k \leq |V(G)|$, a graph is $(k,m)$-knit if whenever $S$ is a set of $k$ vertices of $G$ and $S_1, \ldots, S_t$ is a partition of $S$ into $t \geq m$ non-empty parts, $G$ contains vertex-disjoint connected subgraphs $C_1, \ldots, C_t$ such that $S_i \subseteq V(C_i)$, $1 \leq i \leq t$. Clearly, a $(2k,k)$-knit graph is $k$-linked. In [1], Bollobás and Thomassen proved that if a $k$-connected graph $G$ contains a minor $H$, where $H$ is a graph with minimum degree at least $0.5(|H| + \lfloor 5k/2 \rfloor - 2 - m)$, then $G$ is $(k,m)$-knit. They used this result to show that $22k$-connected graphs are $k$-linked, which is the first linear upper bound of connectivity for a graph to be $k$-linked.

We consider a slightly more general notion than $(k,m)$-knit. For a set $S \subseteq V(G)$ of a graph $G$, the pair $(G,S)$ is knitted if for every partition of $S$ into non-empty subsets $S_1, S_2, \ldots, S_t$, there are disjoint connected subgraphs $C_1, C_2, \ldots, C_t$ in $G$ so that $S_i \subseteq C_i$. A graph $G$ is $\ell$-knitted if $(G,S)$ is knitted for all $S \subseteq V(G)$ with $|S| = \ell$. It is clear that an $\ell$-knitted graph is $(\ell,m)$-knit for all $m \leq \ell$.

In this paper, we give a connectivity condition for a graph to be $\ell$-knitted.

Definition 1. The pair $(A,B)$ is a separation of $G$ if $V(G) = A \cup B$ and there is no edge between $A - B$ and $B - A$. The order of a separation $(A,B)$ is $|A \cap B|$. If $S \subseteq A$, then we say that $(A,B)$ is a separation of $(G,S)$.

We shall prove the following theorem.

Theorem 2. Let $k$ and $\ell$ be positive integers and $S \subseteq V(G)$ with $|S| = \ell < k/9$. If there is no separation of $(G,S)$ of size less than $\ell$, and every vertex in $G - S$ has degree at least $k - 1$, then $(G,S)$ is knitted.

The theorem we will prove, Theorem 7, on edge-density in Section 3 is actually stronger than Theorem 2.

We are now ready to state and prove our result on connectivity of minimal counterexamples to Hadwiger’s Conjecture.

Theorem 3. For all positive integer $k$, every $k$-chromatic minimal (with respect to the minor relation) counterexample to Hadwiger’s Conjecture is $\lceil k/9 \rceil$-connected.

Proof. Assume by contradiction that the statement fails. Then we have a minimal $k$-chromatic graph $G$ that has no $K_k$-minor and is not $k/9$-connected. Take a minimum cutset $S$. Then $|S| < k/9$. Let $A_1$ be a component of $G - S$ and $A_2 = G - S - A_1$. Then both $G[A_1 \cup S]$ and $G[A_2 \cup S]$ have the chromatic number at most $k - 1$.

Let $S_1$ be a maximum independent set in $G[S]$, and let $S_i$ be a maximum independent set in $G\left[S - \bigcup_{j=1}^{i-1}S_j\right]$ for $i \geq 2$. Let $v_1, v_2, \ldots, v_{|S|}$ be the set of vertices in $S$ such that $v_1, \ldots, v_{|S_1|} \in S_1, v_{|S_1| + 1}, \ldots, v_{|S_1| + |S_2|} \in S_2$, and so on. Observe that if we contract each of the subgraph induced by $S_i$ into one vertex, then the resulting graph in $S$ is a clique.

Note that the minimum degree of $G$ is at least $k - 1$, thus each vertex in $A_p$ has at least $k - 1$ neighbors in $A_p \cup S$ for $p \in \{1,2\}$. Note also that a separation in $(A_1 \cup S,S)$ or $(A_2 \cup S,S)$ is a separation in $(G,S)$, thus $(A_1 \cup S,S)$ and $(A_2 \cup S,S)$ have no separation of size less than $\ell$. By Theorem 2, both $(A_1 \cup S,S)$ and $(A_2 \cup S,S)$ are knitted. So there are disjoint connected subgraphs $C_i \subseteq A_1 \cup S$’s and $D_i \subseteq A_2 \cup S$ so that $S_i \subseteq C_i$ and $S_i \subseteq D_i$. Hence we can contract $A_1 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on $S$ is complete. Let $G_1$ be the resulting graph plus $A_2$. Similarly, we can also contract $A_2 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on $S$ is complete (let $G_2$ be the resulting graph plus $A_1$).

Then $\chi(G_1), \chi(G_2) \leq k - 1$ by minimality of $G$. But clearly we can combine the colorings of $G_1$ and $G_2$ to the whole graph $G$ using at most $k - 1$ colors. This is a contradiction. This completes the proof of the theorem. \qed
The rest of the paper is to prove Theorem 2. We will do this in two steps: in the first step (Section 3), we will show a graphs under study either is knitted or has a dense subgraph; in the second step (Section 2), we find a knitted subgraph in the dense subgraph. Note that this approach is very much similar to the one used by Thomas and Wollan [9].

2. Dense graphs are knitted

In this section, we study when a small dense graph contains a knitted subgraph. This is needed in our proof of Theorem 2 in Section 3.

To show a small dense graph is $k$-knitted, we use a result by Faudree et al [3] on $k$-ordered graphs, where a graph is $k$-ordered if for every $k$ vertices of given order, there is a cycle containing the $k$ vertices of the given order. It is clear that a $k$-ordered graph is $k$-knitted. Throughout the paper, we will use $d(x, H)$ to denote the number of neighbors (degree) of $x$ in subgraph $H$ of $G$.

**Theorem 4** (Faudree et al [3]). For every graph $G$ with order $n \geq 2\ell \geq 2$, if $d(x, G) + d(y, G) \geq n + \frac{3n}{2\ell} - 1$ for every pair of non-adjacent vertices $x$ and $y$, then $G$ is $\ell$-ordered.

Note that for $n \geq 5\ell$, Kostochka and Yu [6] showed that a graph $G$ with minimum degree at least $\frac{3n}{2\ell} - 1$ is $\ell$-ordered. Since we do not know if the minimum degree condition still holds for $n < 5\ell$, we are unable to use this less demanding degree conditions in our proof.

**Theorem 5.** Let $\alpha \geq 4.5$. A graph $H$ with minimum degree $\delta(H) \geq \alpha\ell + 1$ and $|V(H)| \leq 2\alpha\ell$ contains an $\ell$-knitted subgraph.

**Proof of Theorem 5.** Assume by contradiction that $H$ is not $\ell$-knitted. Then there is a subset $S \subseteq V(H)$ with $|S| = \ell$, and a partition $S = \bigcup_{i=1}^{t} S_i$ such that we cannot find disjoint connected subgraphs containing $S_i$’s.

We consider partial $(\ell, t)$-knit $C = \bigcup_{i=1}^{t} C_i$, which is a subgraph of $G$ in which $S_i \subseteq C_i$ but $C_i$S are not necessarily connected.

An optimal $(\ell, t)$-knit $C = \bigcup_{i=1}^{t} C_i$ is a partial $(\ell, t)$-knit such that
(a) $|C| \leq \alpha\ell$;
(b) the number of components of $C$ is minimized; and
(c) subject to (a) and (b), $|C|$ is minimized.

We observe that the components in $C$ containing exactly one vertex in $S$ consist of one vertex, and a component with two vertices in $S$ is a path.

We may assume that $S_1 \subseteq C_1$, but $C_1$ is not connected. Then there exists $x, y \in S_1$ such that $x$ and $y$ belong to different components of $C_1$. Note that $H - C \neq \emptyset$, since $d(x, H - C) = d(x) - |C| \geq (\alpha\ell + 1) - \alpha\ell = 1$.

Now we show that for every $u \in H - C$ and for every component $P$ in $C$ with $|V(P) \cap S| \geq 2$, $d(u, P) \leq |V(P) \cap S| + 1$. We actually will give the following more general statement, which might be of independent interest.

**Lemma 1.** Let $W$ be a graph. Let $S'$ be a subset of $V(W)$ with $|S'| \geq 2$, and let $F$ be subtree of $W$ such that $F \supseteq S'$ and all leaves of $F$ belong to $S'$. Let $u \in W - F$, and suppose that $d(u, F) \geq |S'| + 2$. Then $W[V(F) \cup \{u\}]$ contains a subtree $F_0$ with $u \in F_0$ such that $|F_0| < |F|$, $F_0 \supseteq S'$ and all leaves of $F_0$ belong to $S'$.

**Proof.** Let $k = |S'|$. When $k = 2$, $F$ is a path with both leaves in $S'$, then since $d(u, F) \geq 4$, we can replace a segment of $F$ by $u$ to get a smaller subtree $F_0$ so that the leaves of $F_0$ belong to $S'$. So let $k \geq 3$. 

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Now we use induction on $|F|$. Note that $F$ has at least two leaves, and let $u_1, u_2 \in S$ be two of them. For $i = 1, 2$, let $P_i$ be maximal paths such that $u_i \in P_i$ and the subtree $F - V(P_i)$ contains $S' - \{u_i\}$. Note that $P_1 \cap P_2 = \emptyset$. For each $i$, let $x_i$ be the vertex in $F - P_i$ which is adjacent (in $F$) to an endpoint of $P_i$.

Let $i = 1$ or $2$. First assume $d(u, P_i) = 0$. Then by the induction assumption, $W[V(F - P_i) \cup \{u\}]$ contains a subtree $F'$ with $u \in F'$ such that $|F'| < |F - P_i|$, $F' \supseteq (S' - \{u\}) \cup \{x_i\}$ and all leaves of $F'$ belong to $(S' - \{u_i\}) \cup \{x_i\}$. Adding $P_i$ to $F'$, we obtain a desired tree. Next assume $d(u, P_1) = 1$. Then by the induction assumption, $W[V(F - P_1) \cup \{u\}]$ contains a subtree $F'$ with $u \in F'$ such that $|F'| < |F - P_1|$, $F' \supseteq S' - \{u_i\}$ and all leaves of $F'$ belong to $S' - \{u_i\}$. Adding $P_i$ to $F'$, we obtain a desired tree. Thus we may assume $d(u, P_i) \geq 2$ for each $i = 1, 2$. Let $P_i = u_iP_iV_iP_i'x_i'$ so that $x_i'$ is adjacent to $x_i$ and $v_i$ is the only neighbor of $u$ on $u_iP_iV_i$. Then $|V(v'_ix'_i)| \geq 1$. Now $F_0 = (F - \cup_{i=1}^2 V(v'_ix'_i)) \cup \{u\}$ is a subtree (note that $k \geq 3$, so $F_0$ is connected) with desired properties. 

Let $\delta^*$ be the minimum degree of $H - C$. We have the following

**Lemma 2.** $\delta^* \geq (\alpha - 1.5)\ell$.

**Proof.** For every $u \in H - C$,

$$d(u, H - C) = d(u, H) - d(u, C) \geq \delta(H) - d(u, C) \geq \alpha\ell + 1 - d(u, C).$$

So we just need to prove that $d(u, C) \leq 1.5\ell$ for every $u \in H - C$.

Let $P_j, 1 \leq j \leq c_i$, be the components of $C_i$ in which $u$ has neighbors. If $|P_j \cap S| \geq 2$, then by Lemma 1 we have $d(u, P_j) \leq |P_j \cap S| + 1 \leq 3|P_j \cap S|/2$ and, if $|P_j \cap S| = 1$ then $|P_j| = 1$, and hence $d(u, P_j) = |P_j \cap S| \leq 3|P_j \cap S|/2$, which implies

$$d(u, C_i) = \sum_{j=1}^{c_i} d(u, P_j) \leq \sum_{j=1}^{c_i} 3|P_j \cap S|/2 \leq 3|C_i \cap S|/2.$$

Therefore $d(u, C) = \sum_{C_i} d(u, C_i) \leq 1.5|S| = 1.5\ell$, and the lemma is proven. 

**Lemma 3.** The subgraph $H - C$ is connected.

**Proof.** Let $H_1, \ldots, H_p$ with $p \geq 1$ be the components of $H - C$. Then $H_i$ is not $\ell$-knitted, thus not $\ell$-ordered. So by Theorem 4, $2\delta^* < |H_i| + \frac{3\ell - 9}{2}$. Therefore we have

$$|H_i| > (2\alpha - 4.5)\ell + 4.5.$$

If $p \geq 2$, then $|H| \geq |C| + |H_1| + |H_2| > \ell + 2(2\alpha - 4.5)\ell + 9$, that is, $2\alpha\ell > (4\alpha - 8)\ell + 9$. So $(8 - 2\alpha)\ell > 9$, a contradiction to $\alpha \geq 4$.

**Lemma 4.** $|C| \leq \alpha\ell - 5$.

**Proof.** For otherwise, $|H - C| \leq 2\alpha\ell - |C| \leq 2\alpha\ell - (\alpha\ell - 4) = \alpha\ell + 4$. Then $2\delta^* - (|H - C| + \frac{3\ell - 9}{2}) \geq (2\alpha - 3)\ell - (\alpha\ell + 4) - \frac{3\ell - 9}{2} = (\alpha - 4.5)\ell + 0.5 > 0$. By Theorem 4, $H - C$ is $\ell$-ordered, thus $\ell$-knitted, a contradiction.

Let $A = N(x) \cap (H - C)$ and $B = N(y) \cap (H - C)$. Furthermore, let $A' = N(A) \cap (H - C) - A$ and $B' = N(B) \cap (H - C) - B$. Let $D = (H - C) - (A \cup A' \cup B \cup B')$. Then there is no path of length at most 6 from $x$ to $y$ through $A \cup A' \cup D \cup B' \cup B'$, for otherwise, we may get $C'$ by adding this path to $C$. Note that $C'$ has less components than $C$, and $|C'| \leq |C| + 5 \leq (\alpha\ell - 5) + 5 = \alpha\ell$, a contradiction to the assumption that $C$ is optimal. 


Definition 2. A separation \((A, B)\) of \((G, S)\) is rigid if \((G[B], A \cap B)\) is knitted.

For a set \(H \subseteq V(G)\), let \(\rho(H)\) be the number of edges with at least one endpoint in \(H\).

Definition 3. Let \(G\) be a graph and \(S \subseteq V(G)\), and \(\alpha > 1\) be a real number. The pair \((G, S)\) is \(\alpha \ell\)-massed if

\[
\begin{enumerate}
\item \((G, S)\) is \(\alpha \ell\)-massed,
\item \(|S| \leq \ell \) and \((G, S)\) is not knitted,
\item subject to above two, \(|V(G)|\) is minimum,
\item subject to above three, \(\rho(G - S)\) is minimum, and
\item subject to above four, the number of edges of \(G\) with both ends in \(S\) is maximum.
\end{enumerate}
\]

Definition 4. Let \(G\) be a graph and \(S \subseteq V(G)\), and let \(\alpha > 1\) be a real number. The pair \((G, S)\) is \((\alpha, \ell)\)-minimal if

\[
\begin{enumerate}
\item \((G, S)\) is \(\alpha \ell\)-massed,
\item \(|S| \leq \ell \) and \((G, S)\) is not knitted,
\item subject to above two, \(|V(G)|\) is minimum,
\item subject to above three, \(\rho(G - S)\) is minimum, and
\item subject to above four, the number of edges of \(G\) with both ends in \(S\) is maximum.
\end{enumerate}
\]

Theorem 6. Let \(\ell \geq 1\) be an integer and \(\alpha \geq 2\) be a real number. Let \(G\) be a graph and \(S \subseteq V(G)\) such that \((G, S)\) is \((\alpha, \ell)\)-minimal. Then \(G\) has no rigid separation of order at most \(|S|\), and \(G\) has a subgraph \(H\) with \(|V(H)|\) \leq 2\(\alpha \ell\) and minimum degree at least \(\alpha \ell + 1\).

With Theorem 6 and Theorem 5, we can actually obtain the following result, which is a little stronger than Theorem 2.

Theorem 7. Let \(\ell\) be an integer. Let \(G\) be a graph and \(S \subseteq V(G)\) be an \(\ell\)-subset such that \((G, S)\) is \((4.5, \ell)\)-massed. Then \((G, S)\) is knitted.

Proof. Suppose that some \((4.5, \ell)\)-massed graph is not knitted and take such a graph \(G\) so that \((G, S)\) is \((4.5, \ell)\)-minimal. By Theorem 6 and 5, the graph \(G\) has no rigid separation of order at most \(\ell\) and has an \(\ell\)-knitted subgraph \(K\).

If there is \(|S| = \ell\) disjoint paths from \(S\) to \(K\) (we may suppose that each path uses one vertex in \(K\)), then for every partition of \(S\), there is a corresponding partition of the endpoints of the paths in \(K\); since \(K\) is knitted, there are disjoint connected subgraphs in \(K\) containing the parts of the endpoints, thus we have disjoint connected subgraph containing the parts of \(S\).

If there is no \(|S|\) disjoint paths from \(S\) to \(K\), then there is separation \((A, B)\) with \(S \subseteq A, K \subseteq B\) of order at most \(\ell - 1\). We may assume \((A, B)\) is a separation with smallest order. Then there are \(|A \cap B|\) disjoint paths from \(A \cap B\) to \(K\). Similar to the above, for every partition of \(A \cap B\), we have disjoint connected subgraph containing the parts of \(A \cap B\). Since \(G[B, A \cap B]\) is knitted, that is, \((A, B)\) is a rigid separation of order at most \(\ell - 1\), a contradiction.

Proof of Theorem 6. We prove this theorem in the following three claims.

Claim 1. \(G\) has no rigid separation of order at most \(|S|\).
Proof. For otherwise, take a rigid separation \((A, B)\) with minimum \(A\).

We first assume that \(|A \cap B| < |S|\). Let \(G^*[A]\) be the resulting graph from \(G[A]\) by adding all missing edges in \(A \cap B\). Consider \((G^*[A], S)\). If it also satisfies both (i) and (ii), then \((G^*[A], S)\) is knitted, and a knit in \(G^*[A]\) can be easily converted into a knit in \(G\) since \((A, B)\) is a rigid separation. Since \(G\) is \(\alpha\ell\)-massed, \(\rho(B - A) \leq \alpha\ell|B - A|\), hence \(\rho(A - S) > \alpha\ell|A - S| - (\alpha - 0.5)\ell^2\). So it satisfies (i), and thus does not satisfy (ii).

Let \((A', B')\) be a separation of \(G^*[A]\) such that \(S \subseteq A'\) and \(B'\) is minimal. If \(A \cap B \subseteq A'\), then \((A' \cup B, B')\) is a separation in \(G\) violating (ii). So \(A \cap B \not\subseteq A'\). Since \(A \cap B\) forms a cliques, \(A \cap B \subseteq B'\). Consider \((G[B'], A' \cap B')\). The minimality of \(B'\) implies that it satisfies (ii), and \(\rho(B' - A') > \alpha\ell|B' - A'| > \alpha\ell|B' - A'| - 1\) means that it satisfies (i) as well. So \((G[B'], A' \cap B')\) is knitted. Then \((G[B \cup B'], A' \cap B')\) is knitted, which means that \(A' \cap B'\) is a rigid separation of \((G, S)\), a contradiction to the minimality of \(A\).

Now assume that \(|A \cap B| = |S|\). If there exist \(|S|\) disjoint paths from \(S\) to \(A \cap B\) and the paths together with the rigidity of \((A, B)\) show that \((G, S)\) is knitted, a contradiction. So there is a separation \((A'', B'')\) of \((G[A], S)\) of order less than \(|S|\) with \(A \cap B \subseteq B''\). Choose such a separation with minimum \(|A'' \cap B''|\). Then there are \(|A'' \cap B''|\) disjoint paths from \(A'' \cap B''\) to \(A \cap B\), from the rigidity of \((A, B)\) we have \((A'', B \cup B'')\) is a rigid separation of \((G, S)\) with \(|A''| < |A|\), a contradiction to the minimality of \(A\).

\(\Box\)

Claim 2. For every edge \(uv\) with \(v \not\in S\), the vertices \(u\) and \(v\) have at least \(\alpha\ell\) common neighbors.

Proof. Consider the graph \(G' = G/uv\), the resulting graph from \(G\) by contradicting the edge \(uv\). If \((G', S)\) is knitted, then \((G, S)\) is knitted. So \((G', S)\) violates (i) or (ii).

If \((G', S)\) violates (i), then

\[
\rho(G' - S) \leq \alpha\ell|G' - S| - 1 = (\alpha\ell|G - S| - 1 - \alpha\ell < \rho(G - S) - \alpha\ell.
\]

Thus \(u\) and \(v\) have at least \(\alpha\ell\) common neighbors, which gives the difference of sizes of \(G\) and \(G'\).

So we may assume that \((G', S)\) violates (ii). Let \((A', B')\) be a separation of \(G'\) violating (ii) with \(B'\) minimal. By minimality, the pair \((G'[B'], A' \cap B')\) is knitted. So \((A', B')\) is a rigid separation of \((G', S)\) (of order at most \(|S| - 1\)). Note that the separation induces a separation \((A, B)\) in \(G\). If \(\{u, v\} \not\subseteq A \cap B\), then \((A, B)\) is a rigid separation of \((G, S)\) of order at most \(|S| - 1\), which is a contradiction to Claim 1. So we assume that \(u, v \in A \cap B\). Then by minimality of \(B'\), \((G[B], A \cap B)\) is \(\alpha\ell\)-massed thus knitted, so \((A, B)\) is a rigid separation of size at most \(|A' \cap B'| + 1 \leq |S|\), a contradiction to Claim 1 again.

\(\Box\)

Claim 3. Let \(\delta'\) be the minimum degree in \(G\) among the vertices in \(V(G) - S\). Then \(\alpha\ell + 1 \leq \delta' < 2\alpha\ell\).

Proof. We only need to prove that \(\delta' < 2\alpha\ell\). Take an edge \(e = uv\) in \(G\), and consider \(G_1 = G - e\). Then \(G_1\) fails (i) or (ii).

If \(G_1\) fails (ii), then \((G - e, S)\) contains a separation \((A, B)\) with \(|A \cap B| < |S|\). It follows that \(u \in A - B\) and \(v \in B - A\), lest \((A, B)\) is a separation in \((G, S)\) violating (ii). Then \(|N(u) \cap N(v)| \leq |A \cap B| < |S| \leq \ell < \alpha\ell\), a contradiction to Claim 2. So \(G_1\) fails (i), that is, \(\rho(G - S) \leq \alpha\ell|V(G) - S| - 1\).

If \(\delta' \geq 2\alpha\ell\), then

\[
2(\alpha\ell|V(G) - S| - 1) \geq 2\rho(G - S) \geq \sum_{v \in V(G) - S} \deg(v) \geq 2\alpha\ell|V(G) - S|,
\]

a contradiction.

\(\Box\)
Now let \( v \in V(G) - S \) be a vertex with degree \( \delta' \) in \( G \). Let \( H \) be the graph induced by \( v \) and its neighbors. Then \( H \) has at most \( 2\alpha \ell \) vertices, and \( H \) has minimum degree at least \( \alpha \ell + 1 \). □

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REFERENCES


NATIONAL INSTITUTE OF INFORMATICS, 2-1-2 HITOTSUBASHI, CHIYODA-KU, TOKYO 101-8430, JAPAN.
E-mail address: k keniti@nii.ac.jp

DEPARTMENT OF MATHEMATICS, COLLEGE OF WILLIAM AND MARY, WILLIAMSBURG, VA 23185.
E-mail address: gyu@wm.edu

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