Strong list-chromatic index of subcubic graphs

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Abstract

A strong k-edge-coloring of a graph G is an edge-coloring with k colors in which every color class is an induced matching. The strong chromatic index of G, denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k-edge-coloring. In 1985, Erdős and Nešetřil conjectured that $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$, where $\Delta(G)$ is the maximum degree of G. When G is a graph with maximum degree at most 3, the conjecture was verified independently by Andersen and Horák, Qing, and Trotter. In this paper, we consider the list version of strong edge-coloring. In particular, we show that every subcubic graph has strong list-chromatic index at most 11 and every planar subcubic graph has strong list-chromatic index at most 10.

Keywords: Subcubic graphs; Strong choice number; Combinatorial Nullstellensatz

1 Introduction

All graphs in this paper are finite and simple. A *strong k-edge-coloring* of a graph G is a coloring $\phi : E(G) \longrightarrow [k]$ such that if any two edges e_1 and e_2 are either adjacent to each other or adjacent to a common edge, then $\phi(e_1) \neq \phi(e_2)$. In other words, the edges in each color class give an induced matching in the graph; that is, any two vertices belonging to distinct edges with the same color are not adjacent. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k-edge-coloring.

The following conjecture was proposed by Erdős and Nešetřil in 1985 at Prague.

Conjecture 1.1 [5, 6] *If G is a graph with maximum degree $\Delta(G)$, then*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2, & \text{if } \Delta(G) \text{ is even,} \\ \frac{5}{4}\Delta(G)^2 - \frac{1}{2}\Delta(G) + \frac{1}{4}, & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

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Note that there are examples showing that the conjectured upper bound is tight (i.e. blow-ups of a 5-cycle). Andersen [2] and independently Horák, Qing, and Trotter [8] showed that $\chi'_s(G) \leq 10$ for any graph G with $\Delta(G) = 3$, thus settling the first nontrivial case of Conjecture 1.1. Cranston [4] gave an algorithm that uses at most 22 colors for every graph with $\Delta(G) = 4$, which was improved to 21 very recently by Huang, Santana and Yu [15]. When $\Delta(G)$ is sufficiently large, Molloy and Reed [14] proved that $\chi'_s(G) \leq 1.998\Delta(G)^2$. Recently, Bonamy, Perrett, and Postle [3] improved the upper bound to $1.835\Delta(G)^2$.

In this article, we study the list version of strong edge-coloring. For each $e \in E(G)$, let $L(e)$ be the list of available colors of e, and let $L = \{L(e) : e \in E(G)\}$. The graph G is *strongly L-edge-colorable* if there exists a strong edge coloring c of G such that $c(e) \in L(e)$ for every $e \in E(G)$. For a positive integer k, a graph G is *strongly k-edge-choosable* if G is strongly L-edge colorable for every L with $|L(e)| \geq k$ for all $e \in E(G)$. The *strong list-chromatic index*, denoted by $\chi'_{s,l}(G)$, is the minimum positive integer k for which G is strongly k-edge-choosable. Note that $\chi'_s(G) \leq \chi'_{s,l}(G)$ for every graph G.

The probablistic arguments that Molloy-Reed and Bonamy-Perrett-Postle used to give upper bounds of χ'_s on graphs of large $\Delta(G)$ actually also work for the strong list-chromatic index. So we have $\chi'_{s,l}(G) \leq 1.835\Delta(G)^2$ for large $\Delta(G)$. Ma, Miao, Zhu, Zhang and Luo [13] proved that the strong list-chromatic index of a subcubic graph with maximum average degree less than $\frac{15}{7}, \frac{27}{11}, \frac{13}{5}, \frac{36}{13}$ is at most $6, 7, 8, 9$, respectively. More results of this kind can be found in [16].

In this paper, we prove the following result.

Theorem 1.2 *If $\Delta(G) \leq 3$, then $\chi'_{s,l}(G) \leq 11$.*

For planar graphs, we actually can do a little better.

Theorem 1.3 *If G is a subcubic planar graph, then $\chi'_{s,l}(G) \leq 10$.*

Recall that Andersen [2] and Horák, Qing, and Trotter [8] proved that $\chi'_s(G) \leq 10$ if $\Delta(G) \leq 3$. Kostochka et. al. [11] proved that $\chi'_s(G) \leq 9$ under the additional assumption that G is planar. We do not feel that our results are optimal, but it may involve substantial work to improve them.

One of the main tools we use is Hall's Theorem.

Lemma 1.4 (Hall [7]) *Let $A_1, ..., A_n$ be n subsets of a set U. Distinct representatives of $\{A_1, ..., A_n\}$ exist if and only if for all k, $1 \leq k \leq n$ and every choice of subcollection of size k, $\{A_{i_1}, ..., A_{i_k}\}$, we have $|A_{i_1} \bigcup ... \bigcup A_{i_k}| \geq k$.*

Another tool we use is the Combinatorial Nullstellensatz.

Lemma 1.5 (Alon [1], Combinatorial Nullstellensatz) *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree $deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a non-negative integer, and the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in P is non-zero. Then if $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i, i = 1, 2, \ldots, n$, there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.*

We use MATLAB to calculate the coefficients of specific monomials. Let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables, where $n \geq 1$. By $c_p(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n})$, we denote the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in P, where k_i $(1 \leq i \leq n)$ is a non-negative integer. The codes are listed in the Appendix.

2 Basic properties

Consider (G, L) such that G is not L-choosable but any proper subgraph of G is L-choosable. Clearly, G is connected. In this section, we will show that if $|L(e)| \geq 10$ for each $e \in E(G)$, then G is cubic and has no cycles of length at most five.

We first introduce some notation. An *i-vertex* is a vertex of degree i in our graphs. An *i-cycles* is a cycle of length i in graphs. A *partial coloring* of G is a coloring of a proper subgraph of G. Given edges e and e' in G, we say that e *sees* e' if either e and e' are adjacent, or there is another edge e'' adjacent to both e and e'. Note that even if e sees e' in G, e does not necessarily see e' in a proper subgraph of G. Additionally, we will also say that e *sees a color* α if e sees an edge e' of color α. Let ϕ be a partial coloring of G. For $e \in E(G)$, let $C_\phi(e)$ denote the set of colors seen by e, and let $A_\phi(e) = L(e) \setminus C_\phi(e)$. For $v \in V(G)$, $H \subseteq G$, let $d(v, H)$ with respect to v be the minimum of the lengths of the u-v paths of G where $u \in V(H)$.

Lemma 2.1 *G is cubic.*

Proof. By way of contradiction, we assume that $d(v) \leq 2$ for some $v \in V(G)$. By the minimality of G, $G - v$ has an L-coloring ϕ. First let v be a 1-vertex incident with the edge e. Since $|C_\phi(e)| \leq 6$, $|A_\phi(e)| \geq 4$, so e can be colored. Let v be a 2-vertex with incident edges e_1 and e_2. Since $|C_\phi(e_i)| \leq 8$ for $i = 1, 2$, $|A_\phi(e_i)| \geq 2$. So we can color e_1 and e_2 in any order. □

Lemma 2.2 *G has no triangles.*

Proof. Suppose that G contains a triangle: $v_1v_2v_3v_1$ (see Fig.1 (1)). By the minimality of G, let ϕ be an L-coloring of the subgraph $H = G - v_1$. Note that $|A_\phi(e_i)| \geq 3$, for $i = 1, 2$ and $|A_\phi(e_3)| \geq 1$. Then ϕ can be extended to an L-coloring of G by Lemma 1.4. □

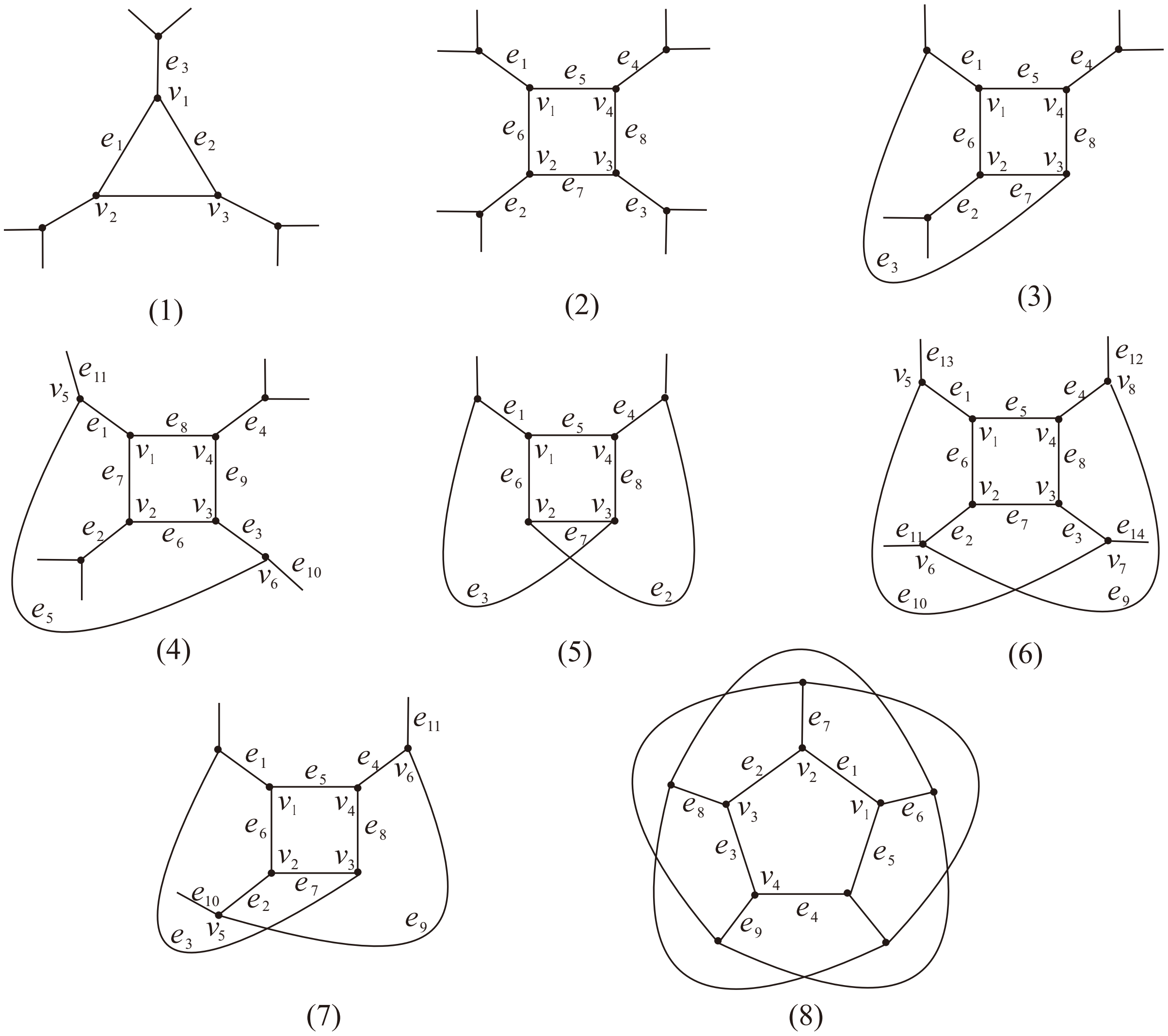


Figure 1: Structures with 3-, 4-, and 5-cycles

Lemma 2.3 *G has no 4-cycles.*

Proof. Suppose that G contains a 4-cycle. By Lemma 2.2, each 4-cycle must be an induced 4-cycle, and we divide all 4-cycles into three classes. The possible local structures about a 4-cycle are shown in Figure 1 (2)-(7). For $2 \le i \le 7$, let H_i be the subgraph of G obtained by removing the vertices with labels in Figure 1 (i) for $2 \le i \le 7$. By minimality of G, H_i has an L-coloring ϕ. Let $A_i = A_\phi(e_i)$ for each labelled edge e_i in the figures.

When e_1 does not see e_3 and e_2 does not see e_4, we consider H_2.

In H_2, we have $|A_i| \ge 4$ for $i = 1, 2, 3, 4$ and $|A_i| \ge 6$ for $i = 5, 6, 7, 8$. By Lemma 1.4, we may assume that for some $I \subseteq [8]$, $|\bigcup_{i\in I} A_i| < |I|$. So $|I| > 6$, and $|I| \in \{7, 8\}$. By symmetry, let $1, 3 \in I$. Then $|A_1 \cup A_3| < 8$, so there exists $\alpha \in A_1 \cap A_3$, and we color e_1 and e_3 with α. Let $A'_i = A_i - \{\alpha\}$. Now for $J \subseteq [8] - \{1, 3\}$, if $|\bigcup_{i\in J} A'_i| < |J|$, then $|J| > 3$, which implies that $J \cap \{5, 6, 7, 8\} \ne \emptyset$, so $|J| \ge 6$. Then $|J| = 6$, $|A'_i| \ge 5$ for $i \in \{5, 6, 7, 8\}$ and $A'_2 \cap A'_4 \ne \emptyset$. Color e_2 and e_4 with $\beta \in A'_2 \cap A'_4$, then we can color e_5, e_6, e_7, e_8 in any order, a contradiction.

When e_1 sees e_3 and e_2 does not see e_4, we consider H_3 and H_4.

Consider H_3. Note that $|A_i| \ge 7$ for $i = 1, 3, 5, 6, 7, 8$ and $|A_i| \ge 4$ for $i = 2, 4$. By Lemma 1.4, we may assume that for some $I \subseteq [8]$, $|\bigcup_{i\in I} A_i| < |I|$. Then $I = [8]$, $|A_i| \ge 7$ for $i \in \{1, 3, 5, 6, 7, 8\}$ and $A_2 \cap A_4 \ne \emptyset$. Color e_2 and e_4 with $\alpha \in A_2 \cap A_4$, and then we can color the rest of edges one by one, a contradiction.

Consider H_4. Note that $|A_i| \ge 4$ for $i = 2, 4, 10, 11$, $|A_5| \ge 6$ and $|A_i| \ge 8$ for $i \in \{1, 3, 6, 7, 8, 9\}$. By Lemma 1.4, we may assume that for some $I \subseteq [11]$, $|\bigcup_{i\in I} A_i| < |I|$. Then $|I \cap \{1, 3, 6, 7, 8, 9\} \ne \emptyset$, so $|I| \ge 9$.

We consider the following cases.

Case 1: $\{6, 11\} \subset I$ (or by symmetry, $\{7, 10\} \subset I$). Then $A_6 \cap A_{11} \ne \emptyset$ and color e_6 and e_{11} with $\alpha \in A_6 \cap A_{11}$. For $i \in [11] - \{6, 11\}$, let $A'_i = A_i - \{\alpha\}$. Then for some $J \subseteq [11] - \{6, 11\}$, $|\bigcup_{i\in J} A'_i| < |J|$. Then $|J| \ge 8$. So at most one of $2, 4, 5, 7, 10$ is not in J.

- $5 \in J$. We may assume that $2 \in J$ as well (or by symmetry, $4 \in J$). Then $A'_2 \cap A'_5 \ne \emptyset$, and we color e_2 and e_5 with $\beta \in A'_2 \cap A'_5$. Let $A''_i = A_i - \{\alpha, \beta\}$ for $i \in [11] - \{2, 5, 6, 11\}$. Then for some $K \subseteq [11] - \{2, 5, 6, 11\}$, $|\bigcup_{i\in K} A''_i| < |K|$. So $K \cap \{1, 3, 7, 8, 9\} \ne \emptyset$ and thus $|K| = 7$. Now we can color e_7 and e_{10} with $\gamma \in A''_7 \cap A''_{10}$, and by Lemma 1.4, color the rest of the edges.

- $5 \notin J$. Then $|J| = 8$, so $A'_2 \cap A'_4 \ne \emptyset$. Color e_2 and e_4 with $\beta \in A'_2 \cap A'_4$. Let $A''_i = A_i - \{\alpha, \beta\}$ for $i \in [11] - \{6, 11, 2, 4\}$. Then for some $K \subseteq [11] - \{2, 4, 6, 11\}$, $|\bigcup_{i\in K} A''_i| < |K|$. So

$K\cap\{1,3,7,8,9\}\neq\emptyset$ and thus $|K|=7$. Now we can color e_7 and e_{10} with $\gamma\in A''_7\cap A''_{10}$, and by Lemma 1.4, color the rest of the edges.

Case 2: $\{6,11\}\not\subset I$ and $\{7,10\}\not\subset I$. Then $|I|=9$ and $2,4,5\in I$. So $A_2\cap A_5\neq\emptyset$. Color e_2 and e_5 with $\alpha\in A_2\cap A_5$. For $i\in[11]-\{2,5\}$, let $A'_i=A_i-\{\alpha\}$. Then for some $J\subseteq[11]-\{2,5\}$, $|\bigcup_{i\in J}A'_i|<|J|$. Then $|J|\geq 8$. Then $\{6,11\}\subset J$ (or by symmetry $\{7,10\}\subset J$), and $A_6\cap A_{11}\neq\emptyset$ and color e_6 and e_{11} with $\beta\in A_6\cap A_{11}$. Let $A''_i=A_i-\{\alpha,\beta\}$ for $i\in[11]-\{2,5,6,11\}$. Then for some $K\subseteq[11]-\{2,5,6,11\}$, $|\bigcup_{i\in K}A''_i|<|K|$. So $K\cap\{1,3,7,8,9\}\neq\emptyset$ and thus $|K|=7$. Now we can color e_7 and e_{10} with $\gamma\in A''_7\cap A''_{10}$, and by Lemma 1.4, color the rest of the edges.

When e_1 sees e_3 and e_2 sees e_4, we consider H_5, H_6 and H_7.

Consider H_5. Note that $|A_i|\geq 7$ for $i=1,2,3,4$ and $|A_i|\geq 8$ for $i=5,6,7,8$. By Lemma 1.4, we may assume that for some $I\subseteq[8]$, $|\bigcup_{i\in I}A_i|<|I|$. Clearly, no such I exists, a contradiction.

Consider H_6. First, $|A_\phi(e_i)|\geq 4$ for $i=11,12,13,14$. We can make it that the colors of e_i are different by Lemma 1.4 for $i=11,12,13,14$. Then we note that $|A_i|\geq 6$ for $i=1,2,3,4$, $|A_i|\geq 8$ for $i=5,6,7,8$ and $|A_i|\geq 4$ for $i=9,10$. By Lemma 1.4, we may assume that for some $I\subseteq[10]$, $|\bigcup_{i\in I}A_i|<|I|$. Then $|I|>8$, so $|I|\in\{9,10\}$. By symmetry, we may assume that $\{4,10\}\subset I$. Then $A_4\cap A_{10}\neq\emptyset$, so color e_4,e_{10} with $\alpha\in A_4\cap A_{10}$. Let $A'_i=A_i-\{\alpha\}$ for $i\in[10]-\{4,10\}$. Then for some $J\subseteq[10]-\{4,10\}$, $|\bigcup_{i\in J}A'_i|<|J|$. It implies that $J\cap\{5,6,7,8\}\neq\emptyset$, thus $J=[10]-\{4,10\}$, and $|A'_i|\geq 7$ for $i\in\{5,6,7,8\}$ and $A'_1\cap A'_9\neq\emptyset$. Color e_1 and e_9 with $\beta\in A'_1\cap A'_9$, then we can color the rest of the edges one by one, a contradiction.

Consider H_7. First, $|A_\phi(e_i)|\geq 4$ for $i=10,11$. We can make it that the color of e_{10} is different from the color of e_{11}. Then we note that $|A_i|\geq 7$ for $i=1,3$, $|A_i|\geq 6$ for $i=2,4$, $|A_i|\geq 8$ for $i=5,6,7,8$ and $|A_\phi(e_9)|\geq 4$. By Lemma 1.4, we may assume that for some $I\subseteq[9]$, $|\bigcup_{i\in I}A_i|<|I|$. Then $I=[9]$ and $A_3\cap A_9\neq\emptyset$. Color e_3 and e_9 with $\alpha\in A_3\cap A_9$, and then we can color the rest of edges one by one, a contradiction. $\square$

Lemma 2.4 *G has no 5-cycles.*

Proof. Suppose that G contains the 5-cycle (see Figure 1 (8)). Then by the minimality of G, there is an L-coloring ϕ of $H=G-\{v_i : i\in[4]\}$. We want to color e_i with a color $s_i\in A_\phi(e_i)$ for $i\in[9]$ such that close ones do not see each other. So we need to find $s_i\in A_\phi(e_i)$ for $i\in[9]$ such that $P(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9)\neq 0$, where

$$P(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)=\frac{\prod\limits_{1\leq k<l\leq 9}(x_k-x_l)}{(x_1-x_9)(x_5-x_8)(x_3-x_6)(x_4-x_7)}.$$

Note that $\deg(P) = 32$, $|A_\phi(e_2)| \geq 6$ and $|A_\phi(e_i)| \geq 5$ for $i \in [9] - \{2\}$. Our MATLAB codes show that $c_P(x_1^4 x_2^5 x_3^4 x_4^4 x_5^3 x_6^3 x_7^3 x_8^3 x_9^3) = -6$. By Lemma 1.5, there exist $s_i \in A_i$ for $i \in [9]$ such that $P(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9) \neq 0$. Note that the polynomial P' of any other 5-cycle in G is a subpolynomial of P, then $P \neq 0$ implies that $P' \neq 0$ as well. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a minimal counterexample. By Lemma 2.1-2.4, the girth of G is at least six. By Euler's formula, $\sum_{v \in V(G)}(2d(v) - 6) + \sum_{f \in F(G)}(d(f) - 6) = -12$. It follows that the minimum degree of G is at most two, a contradiction to Lemma 2.1 that G is 3-regular. □

3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Let (G, L) be a minimal counterexample, where $|L(e)| \geq 11$ for each $e \in E(G)$. Without loss of generality, we assume $|L(e)| = 11$. By Lemma 2.1-2.4, G is 3-regular and the girth of G is at least six. Let $v \in V(G)$ with $N(v) = \{v_1, v_2, v_3\}$, and let $N(v_i) - \{v\} = \{w_i, w_i'\}$ for $i \in [3]$. Then w_1, w_2, w_3 form an independent set.

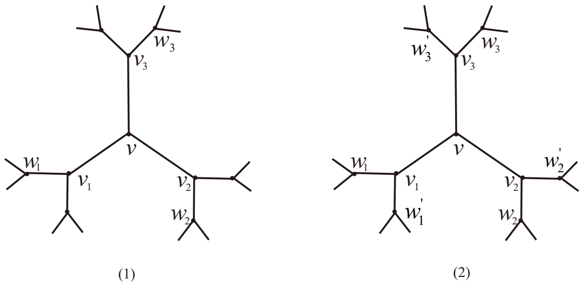
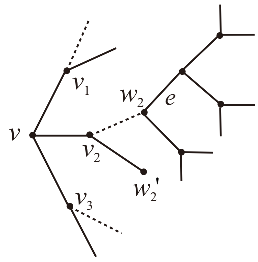


Figure 2:

Lemma 3.1 *Each precoloring of v_1w_1, v_2w_2, v_3w_3 from their lists can be extended to an L-coloring of $H = G - v$.*

Proof. Order the edges in G with respect to the distance from v, that is, if edge e precedes edge f, then $d(v, e) \geq d(v, f)$, where $d(v, e), d(v, f)$ are the distance from v to the edges e and f, respectively. Then the last three edges in the list are vv_1, vv_2 and vv_3, and v_iw_i, v_iw_i' for $i \in [3]$ precede them. Color the edges in the list from the first to the last greedily. For each $e = xy$ in the list with $d(v, x) \geq d(v, y) \geq 1$, y is adjacent to some vertex z with $d(v, z) < d(v, y)$. So the three edges incident with z are after the edge e in the list. Clearly, at least two of the three edges at z



The dashed lines means the edges precolored. It is clear that e sees at most 10 different colors.

Figure 3:

are not precolored, thus e sees at least two uncolored edges in G (Figure 3). So e sees at most 10 different colors, and thus can be colored. $\square$

For $i \in [3]$, let $B_i = L(v_iw_i) \cup L(v_iw_i')$ and $L_i = L(vv_i)$. We will prove Theorem 1.2 through a series of claims.

(1) $B_1 \cap B_2 \cap B_3 = \emptyset$.

For otherwise, precolor v_iw_i (or v_iw_i') with $\alpha \in B_1 \cap B_2 \cap B_3$, which can be extended to an L-coloring of H by Lemma 3.1. Now vv_i for $i \in [3]$ sees at most 8 different colors, so $|A(vv_i)| \geq 3$. So vv_1, vv_2, vv_3 can be colored in the order.

(2) For any $i, j \in [3]$ with $i \neq j$, $B_i \cap B_j \subseteq L_1 \cap L_2 \cap L_3$.

Suppose that for some $i, j \in [3]$ with $i \neq j$, there exists $\alpha \in (B_i \cap B_j) - (L_1 \cap L_2 \cap L_3)$. It means α must not belong to the one of L_1, L_2, L_3. We assume $\alpha \in (B_2 \cap B_3) - L_1$. Without loss of generality, assume $\alpha \in L(v_2w_2) \cap L(v_3w_3) - L_1$. Precolor v_2w_2, v_3w_3 with α and by Lemma 3.1, we can extend it to an L-coloring ϕ of H. Then $|A_\phi(vv_1)| \geq 3$, $|A_\phi(vv_2)| \geq 2$, $|A_\phi(vv_3)| \geq 2$, and we can color vv_2, vv_3, vv_1 by Lemma 1.4.

(3) For some $i, j \in [3]$ with $i \neq j$, $L_i \cap L_j \neq \emptyset$.

For otherwise, in an L-coloring of H, each of vv_1, vv_2, vv_3 has an available color and the colors are distinct, so they could be colored.

We may assume that $L_2 \cap L_3 \neq \emptyset$.

(4)$|\bigcup_{i=1}^3 B_i| \geq |\bigcup_{i=1}^3 L_i| + |L_1 \cap L_2 \cap L_3|$.

By (1), $B_1 \cap B_2, B_2 \cap B_3, B_3 \cap B_1$ are disjoint, and by (2), are subsets of $L_1 \cap L_2 \cap L_3$. So

$$|\bigcup_{i=1}^{3} B_i| = \sum_{i=1}^{3} |B_i| - \sum_{i,j\in[3]} |B_i \cap B_j| + |\bigcap_{i=1}^{3} B_i| \geq \sum_{i=1}^{3} |B_i| - |\bigcap_{i=1}^{3} L_i| = 33 - |\bigcap_{i=1}^{3} L_i|.$$

On the other hand,

$$|\bigcup_{i=1}^{3} L_i| = \sum_{i=1}^{3} |L_i| - \sum_{i,j\in[3]} |L_i \bigcap L_j| + |\bigcap_{i=1}^{3} L_i| \leq 33 - 2|\bigcap_{i=1}^{3} L_i|.$$

Therefore, $|\bigcup_{i=1}^{3} B_i| \geq |\bigcup_{i=1}^{3} L_i| + |L_1 \cap L_2 \cap L_3|$.

(5) For some $i, j \in [3]$ with $i \neq j$, $B_i \bigcap B_j \neq \emptyset$.

For otherwise, $|B_i| \geq 11$ for $i = 1, 2, 3$ and $|\bigcup_{i=1}^{3} B_i| \geq 33$. Since $L_2 \cap L_3 \neq \emptyset$, we have $|\bigcup_{i=1}^{3} L_i| \leq 32$. So there exists $\alpha \in (B_1 \cup B_2 \cup B_3) - (L_1 \cup L_2 \cup L_3)$. Assume $\alpha \in L(v_1w_1) \subset B_1$. Since $|B_2 \cup B_3| \geq 22$ and $|L_2 \cup L_3| \leq 21$, there exists $\beta \in (B_2 \cup B_3) - (L_2 \cup L_3)$, and we may assume $\beta \in L(v_2w_2)$. Now we precolor v_1w_1 with α and v_2w_2 with β, and by Lemma 3.1, extend it to an L-coloring ϕ of H. Now $|A_\phi(vv_1)| \geq 2$, $|A_\phi(vv_2)| \geq 3$, $|A_\phi(vv_3)| \geq 3$, we can color vv_3, vv_2, vv_1 by Lemma 1.4.

By (5) and (2), $|L_1 \cap L_2 \cap L_3| \geq 1$, so by (4), there exists $\alpha \in \bigcup_{i=1}^{3} B_i - \bigcup_{i=1}^{3} L_i$. Assume $\alpha \in L(v_1w_1) \subseteq B_1$.

Precolor v_1w_1 with α.

- $B_2 \cap B_3 \neq \emptyset$.

  Let $\beta \in L(v_2w_2) \cap L(v_3w_3)$. Precolor v_2w_2, v_3w_3 with β. By Lemma 3.1, this precoloring can be extended to an L-coloring ϕ of H. Note that for $i \in [3]$, $|A_\phi(vv_i)| \geq 3$, we can color vv_1, vv_2, vv_3 in the order.

- $B_2 \cap B_3 = \emptyset$.

  Then $|B_2 \cup B_3| = 22 - |B_2 \cap B_3| = 22 > 22 - |L(e_2) \cap L(e_3)| = |L(e_2) \cup L(e_3)|$. So there exists $\beta \in (B_2 \cup B_3) - (L_2 \cup L_3)$. Suppose that $\beta \in L(v_3w_3)$ without loss of generality. Precolor v_3w_3 with β, by Lemma 3.1, this precoloring can be extended to an L-coloring ϕ of H. Note that $|A_\phi(vv_1)| \geq 2, |A_\phi(vv_2)| \geq 3, |A_\phi(vv_3)| \geq 3$, and we can color vv_1, vv_2, vv_3 in the order. □

4 Final discussion

As we mentioned in the introduction, one may try to improve our results by one, which, if true, would be optimal. But this may not be easy, especially for subcubic planar graphs.

Here is another related question. A graph is *chromatic-choosable* if its chromatic number equals to its list chromatic number. It is an interesting problem to find graphs that are chromatic-choosable. Zhu asked whether there exists a constant integer k such that the k-th power G^k is chromatic-choosable for every graph G. Kim, Kwon, and Park [9] answered this question negatively. Moreover, for any fixed k they showed that there are graphs G such that the value $\chi_l(G^k) - \chi(G^k)$ can be arbitrarily large.

We know $\chi'_{s,l}(G)$ is the list chromatic number of the square of the line graph of G. Kostochka and Woodall [12] asked whether G^2 is chromatic-choosable for every graph. Kim and Park [10] solved the conjecture in the negative by finding a family of graphs G whose squares are complete multipartite graphs with partite sets of unbounded size.

Question 4.1 *Is G^2 chromatic-choosable for every line graph G?*

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References

[1] N. Alon, Combinatorial Nullstellensatz, Combinatorics Probability and Computing 8 (1999) 7–29.

[2] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, Discrete Mathematics 108 (1992) 231–252.

[3] M. Bonamy, T. Perrett, L. Postle, Colouring graphs with sparse neighbourhoods, Bounds and Applications (Submitted).

[4] D. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors, Discrete Mathematics 306 (2006) 2772–2778.

[5] P. Erdős, Problems and results in combinatorial analysis and graph theory, Discrete Mathematics 72 (1988) 81–92.

[6] P. Erdős, J. Nešetřil, [Problem], in: G. Halász, V. T. Sós(Eds.), Irregularities of Partitions, Springer, Berlin, 1989, 161–349.

[7] P. Hall, On representatives of subsets, Journal of the Lond Mathematical Society 10(1935) 26–30.

[8] P. Horák, H. Qing, W. T. Trotter, Induced matchings in cubic graphs, Journal of Graph Theory 17 (1993) 151–160.

[9] S. J. Kim, Y. S. Kwon, B. Park, Chromatic-choosability of the power of graphs, Discrete Applied Mathematics 180 (2015) 120–125.

[10] S. J. Kim, B. Park, Counterexamples of list square coloring conjecture, Journal of Graph Theory 78 (2015) 239–247.

[11] A. V. Kostochka, X. Li, W. Ruksasakchai, M. Santana, T. Wang, G. Yu, Strong chromatic index of subcubic planar multigraphs, European Journal of Combinatorics 51 (2016) 380–397.

[12] A. V. Kostochka, D. R. Woodall, Choosability conjectures and multicircuits, Discrete Mathematics 240 (2001) 123–143.

[13] H. Ma, Z. Miao, H. Zhu, J. Zhang, R. Luo, Strong list edge coloring of subcubic graphs, Mathematical Problems in Engineering 2013 (2013) 6 pages.

[14] M. Molloy, B. Reed, A bound on the strong chromatic index of a graph, Journal of Combinatorial Theory Series B 69 (1997) 103–109.

[15] M. Huang, M. Santana, G. Yu, Strong chromatic index of graphs with maximum degree four, submitted.

[16] H. Zhu, Z. Miao, On strong list edge coloring of subcubic graphs, Discrete Mathematics 333 (2014) 6–13.

Appendix

Note that if $P(x_1, x_2, \cdots, x_m)$ is a polynomial with $\deg(P) = n$, and $k_1, k_2, \cdots, k_m$ are non-negative integers with $\sum_{i=1}^{m} k_i = n$. Let $c_P(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m})$ be the coefficient of monomial $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ in P. Then

$$\frac{\partial^n P}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_m^{k_m}} = c_P(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}) \prod_{i=1}^{m} k_i!.$$

```
%input
syms x1 x2 x3 x4 x5 x6 x7 x8 x9
%Lemma 2.4
Q=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x6)*(x1-x7)*(x1-x8)*(x2-x3)*(x2-x4)*(x2
    -x5)*(x2-x6)*(x2-x7)*(x2-x8)*(x2-x9)*(x3-x4)*(x3-x5)*(x3-x7)*(x3-x8)*(x3-
    x9)*(x4-x5)*(x4-x6)*(x4-x8)*(x4-x9)*(x5-x6)*(x5-x7)*(x5-x9)*(x6-x7)*(x6-x8
    )*(x6-x9)*(x7-x8)*(x7-x9)*(x8-x9);
C=diff(diff(diff(diff(diff(diff(diff(diff(diff(Q,x1,4),x2,5),x3,4),x4,4),x5
    ,3),x6,3),x7,3),x8,3),x9,3)/factorial(4)/factorial(5)/factorial(4)/
    factorial(4)/factorial(3)/factorial(3)/factorial(3)/factorial(3)/factorial
    (3)
%output
C=-6
```