PLANAR GRAPHS WITH GIRTH AT LEAST 5 ARE $(3,4)$-COLORABLE

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Abstract. A graph is $(d_1,\dots,d_k)$-*colorable* if its vertex set can be partitioned into k nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i\in\{1,\dots,k\}$. It is known that for each pair (d_1,d_2), there exists a planar graph with girth 4 that is not (d_1,d_2)-colorable. This sparked the interest in finding the pairs (d_1,d_2) such that planar graphs with girth at least 5 are (d_1,d_2)-colorable. Given $d_1\le d_2$, it is known that planar graphs with girth at least 5 are (d_1,d_2)-colorable if either $d_1\ge 2$ and $d_1+d_2\ge 8$ or $d_1=1$ and $d_2\ge 10$. We improve an aforementioned result by providing the first pair (d_1,d_2) in the literature satisfying $d_1+d_2\le 7$ where planar graphs with girth at least 5 are (d_1,d_2)-colorable. Namely, we prove that planar graphs with girth at least 5 are $(3,4)$-colorable.

1. Introduction

All graphs in this paper are finite and simple, which means no loops and no multiple edges. Given a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. A graph is $(d_1,\dots,d_k)$-*colorable* if its vertex set can be partitioned into k nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i\in\{1,\dots,k\}$. This notion is known as *improper coloring*, or *defective coloring*, and has recently attracted much attention. Improper coloring is a relaxation of the traditional proper coloring, however, it also opens up an opportunity to gain refined information on partitioning the graph compared to the traditional proper coloring.

The Four Color Theorem [?, ?] states that the vertex set of a planar graph can be partitioned into four independent sets; this means that every planar graph is $(0,0,0,0)$-colorable since an independent set induces a graph with maximum degree at most 0. A natural question to ask is what happens when we try to partition the vertex set of a planar graph into fewer parts. Already in 1986, Cowen, Cowen, and Woodall [?] proved that a planar graph is $(2,2,2)$-colorable. The previous result is sharp since Eaton and Hull [?] and independently Škrekovski [?] both acknowledged the existence of a planar graph that is not $(1,k,l)$-colorable for any given k and l; for an explicit construction see [?]. Hence, improper coloring of a planar graph with no restriction is completely solved.

Since sparser graphs are easier to color, a natural direction of research is to consider sparse planar graphs, and a popular sparsity condition is imposing a restriction on girth. Grötzsch's theorem [?] states that a planar graph with girth at least 4 is $(0,0,0)$-colorable. Therefore it only remains to consider partitioning the vertex set of a planar graph into two parts. Moreover, since there exists a

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planar graph with girth 4 that is not (d_1, d_2)-colorable for each pair (d_1, d_2) (see [?] for an explicit construction), there has been a considerable amount of research towards improper coloring planar graphs with girth at least 5. For various results regarding improper coloring planar graphs with girth at least 6 or other sparse graphs that are not necessarily planar, see [?, ?, ?, ?, ?, ?]. Similar research has also been done for graphs on surfaces as well [?].

In this paper, we focus on planar graphs with girth at least 5. Škrekovski [?] showed that planar graphs with girth at least 5 are $(4, 4)$-colorable and Borodin and Kostochka [?] proved a result that implies planar graphs with girth at least 5 are $(2, 6)$-colorable. Answering a question by Raspaud, Choi and Raspaud [?] proved that planar graphs with girth at least 5 are $(3, 5)$-colorable. Recently, Choi et al. [?] proved that planar graphs with girth at least 5 are $(1, 10)$-colorable, which answered a question by Montassier and Ochem [?] in the affirmative. By a construction of Borodin et al. [?], it is also known that planar graphs with girth at least 5 (even 6) are not necessarily $(0, d)$-colorable for an arbitrary d. As a conclusion, there are only finitely many pairs (d_1, d_2) that are unknown for which planar graphs with girth at least 5 are (d_1, d_2)-colorable. To sum up, all previous knowledge about improper coloring planar graphs with girth at least 5 are the following:

Theorem 1.1. *Given $d_1 \le d_2$, planar graphs with girth at least 5 are (d_1, d_2)-colorable if*

(1) $d_1 \ge 2$ and $d_1 + d_2 \ge 8$ [?, ?, ?]

(2) $d_1 = 1$ and $d_2 \ge 10$ [?]

In this paper, we prove the following theorem, which reveals the first pair (d_1, d_2) satisfying $d_1 + d_2 \le 7$ where planar graphs with girth at least 5 are (d_1, d_2)-colorable.

{*thm:main*}

Theorem 1.2. *Planar graphs with girth at least 5 are $(3, 4)$-colorable.*

The above theorem also improves the best known answer to the following question, which was explicitly stated in [?]:

Question 1.3 ([?])**.** *What is the minimum d_2^3 such that planar graphs with girth at least 5 are $(3, d_2^3)$-colorable?*

Since Montassier and Ochem [?] constructed a planar graph with girth 5 that is not $(3, 1)$-colorable, along with Theorem 1.2, this shows that $d_2^3 \in \{2, 3, 4\}$. Theorem [?] is an improvement to the previously best known bound, which was by Choi and Raspaud [?]. It would be remarkable to determine the exact value of d_2^3.

Section 2 will reveal some structural properties of a minimum counterexample to Theorem 1.2. In Section 3, we will show that a minimum counterexample to Theorem 1.2 cannot exist via discharging, hence proving the theorem.

We end the introduction with some definitions that will be used throughout the paper. Throughout the paper, let G be a counterexample to Theorem 1.2 with the minimum number of 3^+-vertices, and subject to that choose one with the minimum number of edges. It is easy to see that G must be connected and there are no 1-vertices in G. From now on, given a (partially) $(3, 4)$-colored graph, let i be the color of the color class where maximum degree i is allowed for $i \in \{3, 4\}$. We say a vertex with color i is *i-saturated* if it already has i neighbors of the same color. A vertex is *saturated* if it is either 3-saturated or 4-saturated.

A *d-vertex*, a *d^--vertex*, and a *d^+-vertex* is a vertex of degree d, at most d, and at least d, respectively. A *d-neighbor* of a vertex is a neighbor that is a d-vertex. A d-vertex is a *poor d-vertex* (or *dp-vertex*) and a *semi-poor d-vertex* (or *ds-vertex*) if it has $d-1$ and $d-2$, respectively, 2-neighbors; otherwise, it is called a *rich vertex* (or *dr-vertex*). A *kr^+-vertex* is a rich k^+-vertex. Other analogous terms are defined accordingly. An edge uv is a *heavy edge* if both u and v are 5^+-vertices, and neither u nor v is a $5p$-, $5s$-, or $6p$-vertex.

2. Structural Lemmas

In this section, we reveal useful structural properties of G.

{lem:edge}

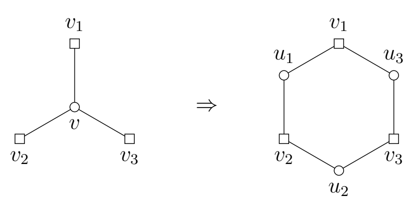
Lemma 2.1. *Every edge xy of G has an endpoint with degree at least 5.*

Proof. Suppose to the contrary that x and y are both 4^--vertices. Since $G - xy$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3,4)$-coloring φ of $G - xy$. If either $\varphi(x) \neq \varphi(y)$ or $\varphi(x) = \varphi(y) = 4$, then φ is also a $(3,4)$-coloring of G. Otherwise, $\varphi(x) = \varphi(y) = 3$, and at least one of x, y is 3-saturated in $G - xy$. For one 3-saturated vertex in $\{x, y\}$, we may recolor it with the color 4, since all of its neighbors have color 3 in G. In all cases we end up with a $(3,4)$-coloring of G, which is a contradiction. □

{lem:3vx}

Lemma 2.2. *There is no 3-vertex in G.*

Proof. Suppose to the contrary that v is a 3-vertex of G with neighbors v_1, v_2, v_3. By Lemma 2.1, we know that v_1, v_2, v_3 are 5^+-vertices. Obtain a graph H from $G - v$ by adding paths $v_1u_1v_2, v_2u_2v_3, v_3u_3v_1$ of length two between the neighbors of v. See Figure 1 for an illustration. Note that H is planar and still has girth at least 5 since the pairwise distance between v_1, v_2, v_3 did not change. Since H has fewer 3^+-vertices than G, there is a $(3,4)$-coloring φ of H.



{fig:3vx}

Figure 1. Obtaining H from G in Lemma 2.2.

Without loss of generality, we may assume $\varphi(u_1) = \varphi(u_2)$. Since each of v_1, v_2, v_3 has a neighbor in $\{u_1, u_2\}$, using the color $\varphi(u_1)$ on v gives a $(3,4)$-coloring of G, which is a contradiction. □

rtex-degree}

Lemma 2.3. *If v is an 8^--vertex of G, then in every $(3,4)$-coloring of $G - v$, v has a saturated neighbor in $G - v$ that cannot be recolored. In particular,*

(i) if $d(v) = 2$, then for each $i \in \{3,4\}$, v has an i-saturated $(i+2)^+$-neighbor u that cannot be recolored. Moreover, if u is an 8^--vertex, then u has a j-saturated $(j+2)^+$-neighbor where $\{i, j\} = \{3,4\}$.

(ii) if $d(v) \in \{4,5\}$, then v has a 4-saturated neighbor that is either a 9^+-vertex or a $6s^+$-vertex.

(iii) if $d(v) \in \{6,7,8\}$, then v has a saturated neighbor that is either a 9^+-vertex or a $5s^+$-vertex.

Proof. Since $G - v$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there exists a $(3,4)$-coloring φ of $G - v$. Note that for each $i \in \{3,4\}$, since letting $\varphi(v) = i$ cannot be a $(3,4)$-coloring of G, v has either an i-saturated neighbor or $i+1$ neighbors with the color i. Since v is an 8^--vertex, v cannot have both four neighbors of color 3 and five neighbors of color 4. Let $j \in \{3, 4\}$ such that v has at most j neighbors with color j, one of which is j-saturated. If every j-saturated neighbor of v can be recolored, then we can color v with j, a contradiction. Hence, v must have at least one j-saturated neighbor that cannot be recolored.

Let u be a non-recolorable j-saturated neighbor of v and let $\{i, j\} = \{3, 4\}$. We know u is a $(j+2)^+$-vertex, since it is adjacent to v, j neighbors colored with j, and at least one neighbor x colored with i (since u cannot be recolored with i). Moreover, if $d(u) \le 8$, then x must be i-saturated. In particular,

- (i) if $d(v) = 2$, then v has both a non-recolorable 3-saturated neighbor and a non-recolorable 4-saturated neighbor. For $j \in \{3, 4\}$, the j-saturated neighbor has degree at least $j + 2$, and if its degree is at most 8, then it has an i-sautarated neighbor of degree at least $i + 2$, where $\{i, j\} = \{3, 4\}$.
- (ii) if $d(v) \in \{4, 5\}$, then v must have a non-recolorable 4-saturated neighbor u. So u is either a 9^+-vertex or a $6s^+$-vertex.
- (iii) if $d(v) \in \{6, 7, 8\}$, then u must be either a 9^+-vertex or a $5s^+$-vertex.

This finishes the proof of this lemma. □

{lem:6cycle}

Lemma 2.4. *Let C be a 6-cycle $u_1u_2u_3u_4u_5u_6$ of G.*

- (a) *If C contains three 2-vertices and a 5-vertex, then the other two vertices are 7^+-vertices.*
- (b) *If C contains exactly two 2-vertices, then C contains at most two $5p$-vertices. Moreover,*
  - $(b1)$ *if C contains exactly one $5p$-vertex, then it contains at most two of $5s$-vertices and $6p$-vertices;*
  - $(b2)$ *if C contains two $5p$-vertices, then either $C = F_{6a}$ (see Figure 2) or it contains neither $5s$-vertices nor $6p$-vertices.*
- (c) *If C contains exactly one 2-vertex, then it contains at most one $5p$-vertex. Moreover,*
  - $(c1)$ *if C contains exactly one $5p$-vertex, then it contains at most two of $5s$-vertices and $6p$-vertices;*
  - $(c2)$ *if C contains no $5p$-vertices, then it contains at most four of $5s$-vertices and $6p$-vertices.*
- (d) *If C contains no 2-vertex, then it contains no poor vertices and at most four $5s$-vertices.*

Proof. Note that by Lemma 2.1, no two 2-vertices are adjacent to each other. We will show that if C is not one of the above, then we can obtain a $(3, 4)$-coloring of G, which is a contradiction.

(a): Let u_1, u_3, u_5 be the 2-vertices and let u_4 be a 5-vertex of C. By Lemma 2.3 (i), both u_2 and u_6 are 6^+-vertices, so without loss of generality, suppose to the contrary that u_6 is a 6-vertex. Since $G - u_5$ is a graph with fewer edges than G and the number of 3^+-vertices does not increase, there is a $(3, 4)$-coloring φ of $G - u_5$. By Lemma 2.3 (i), we know u_4 is 3-saturated and has a 4-saturated 6^+-neighbor and u_6 is 4-saturated and has a 3-saturated 5^+-neighbor. Hence, $\varphi(u_3) = 3$ and $\varphi(u_1) = 4$.

If $\varphi(u_2) = 3$, then recolor u_3 with 4 and color u_5 with 3 to obtain a $(3, 4)$-coloring of G. If $\varphi(u_2) = 4$, then recolor u_1 with 3 and color u_5 with 4 to obtain a $(3, 4)$-coloring of G.

(b): Note that each $5p$-vertex on C must have a 2-neighbor on C, and by Lemma 2.3 (i), each 2-vertex has at most one $5p$-neighbor. So C contains at most two $5p$-vertices because it has exactly two 2-vertices.

(b1) Assume u_1 is the unique $5p$-vertex on C. By Lemma 2.3 (ii), none of u_2, u_6 is a $5s$- or $6p$-vertex. If u_4 is neither a $5s$-vertex nor a $6p$-vertex, then C contains at most two of $5s$-vertices and $6p$-vertices. If u_4 is a $6p$-vertex, then either u_3 or u_5 is 2-vertex, so again C contains at most two of $5s$-vertices and $6p$-vertices. If u_4 is a $5s$-vertex, then by Lemma 2.3 (ii), one of u_3 and u_5

must be either a $6s^+$-vertex or a 9^+-vertex. Therefore, C contains at most two of $5s$-vertices and $6p$-vertices.

(b2) Now assume C contains two $5p$-vertices. Observe that if u_1, u_4 are the two $5p$-vertices on C, then by Lemma 2.3 (ii), none of u_2, u_3, u_5, u_6 is a $5s$-vertex or a $6p$-vertex, as claimed. Therefore, we may assume that u_1, u_3 are the two $5p$-vertices on C.

Note that u_2 cannot be a 2-vertex by Lemma 2.3 (i). So both u_4 and u_6 are 2-vertices. By Lemma 2.3 (i) and (ii), both u_2 and u_5 are 6^+-vertices. We may assume that u_5 is a $6p$-vertex, for otherwise C contains neither $5s$-verices nor $6p$-vertices. Assume that C is not a special 6-face F_{6a}, which implies that u_2 is a 6-vertex. By Lemma 2.3 (i), in a $(3,4)$-coloring φ of $G - u_6$, we know u_1 is 3-saturated and u_5 is 4-saturated and both are non-recolorable. It follows that u_2 is 4-saturated, u_4 is colored with 4 and non-recolorable, and furthermore u_3 is 3-saturated. Now we can recolor u_4, u_3, u_2, u_1 with $3, 4, 3, 4$ respectively, and color u_6 with 3 to obtain a $(3,4)$-coloring of G.

(c): Let u_1 be the unique 2-vertex on C. A $5p$-vertex must have a 2-neighbor on C, and by Lemma 2.3 (i), a 2-vertex has at most one $5p$-neighbor, so C contains at most one $5p$-vertex.

(c1) Assume C has one $5p$-vertex u_2. By Lemma 2.3 (i) and (ii), u_6 cannot be a 5-vertex, and u_3 cannot be a $5s$-vertex or a $6p$-vertex. If u_6 is not a $6p$-vertex, then C has at most two of $5s$-vertices and $6p$-vertices. If u_6 is a $6p$-vertex, then u_4 and u_5 cannot be both $5s$-vertices by Lemma 2.3 (ii). Note that either u_4 or u_5 cannot be $6p$-vertices since C has only one 2-vertex u_1.

(c2) Now assume C contains no $5p$-vertices. Consider three consecutive vertices u_{i-1}, u_i, u_{i+1} on C. If u_i is a $6p$-vertex, then either u_{i-1} or u_{i+1} must be a 2-vertex. If u_i is a $5s$-vertex, then by Lemma 2.3 (ii), either u_{i-1} or u_{i+1} is a 9^+-vertex or a $6s^+$-vertex. Therefore, C contains at most four of $5s$-vertices and $6p$-vertices.

(d): If C contains no 2-vertex, then it contains neither a $5p$-vertex nor a $6p$-vertex. By Lemma 2.3 (ii), a 5-vertex must have a 6^+-neighbor, so the two 3^+-neighbors of a $5s$-vertex cannot be both $5s$-vertices. Therefore, C contains at most four $5s$-vertices. □

{lem:F2}

Lemma 2.5. *If F_{6b} is a 6-cycle with three 2-vertices and three $6p$-vertices (see Figure 2), then F_{6b} cannot share an edge with a 5-cycle with two 2-vertices.*

Proof. Let $u_1 \dots u_6$ be F_{6b} with three 2-vertices u_1, u_3, u_5 and three $6p$-vertices. Note that two 2-vertices cannot be adjacent to each other by Lemma 2.1. Without loss of generality, suppose to the contrary that $u_6u_1u_2v_1v_2$ is a 5-cycle sharing an edge with C. Note that C and $u_6u_1u_2v_1v_2$ cannot share exactly one edge. By symmetry, we may assume that v_1 is a 2-vertex.

Since $G - u_1$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3,4)$-coloring φ of $G - u_1$. By Lemma 2.3 (i), both u_2 and u_6 are non-recolorable and one of u_2 and u_6 is 3-saturated and the other is 4-saturated.

First assume u_6 is 3-saturated and u_2 is 4-saturated. Since u_2 is a 6-vertex, by Lemma 2.3 (i), u_2 must have exactly one 3-saturated neighbor and all other neighbors are colored with the color 4. In particular, $\varphi(v_1) = 4$. Also, by Lemma 2.3 (i), u_6 has a 4-saturated neighbor, which must be v_2. Hence, we can recolor v_1 with the color 3 and color u_1 with the color 4 to obtain a $(3,4)$-coloring of G, which is a contradiction.

Now assume u_6 is 4-saturated and u_2 is 3-saturated. By Lemma 2.3 (i), u_6 must have a 3-saturated neighbor, which must be v_2, and all other neighbors are colored with the color 4. In particular, $\varphi(u_5) = 4$. Also, by Lemma 2.3, we know u_2 must have a 4-saturated neighbor, which is neither u_3 nor v_1. If $\varphi(v_1) = 3$, then we can recolor v_1 with the color 4 and color u_1 with the color 3 to obtain a $(3,4)$-coloring of G, which is a contradiction. Therefore, $\varphi(v_1) = 4$, which further implies that $\varphi(u_3) = 3$. Now, if we can recolor u_3 with the color 4, then we can color u_1 with the color 3 to obtain a $(3, 4)$-coloring of G, which is a contradiction. Hence, u_4 must be 4-saturated, and in particular $\varphi(u_4) = 4$. Finally, we can recolor u_5 with the color 3 and color u_1 with 4 to obtain a $(3, 4)$-coloring of G, which is a contradiction. □

{lem:5-face-

Lemma 2.6. *If C is a 5-cycle $u_1 \dots u_5$ with exactly one 2-vertex u_1, then either*

- *C contains at most two of 5p-, 5s-, and 6p-vertices, or*
- *C is a special 5-face F_{5c} or F_{5d} in Figure 2.*

Proof. Assume that C contains at least three $5p$-, $5s$-, and $6p$-vertices. By symmetry, we may assume that u_3 is a $5s$-vertex. Note that u_2 is not a $5p$-vertex, by Lemma 2.3. Let u be a 2-neighbor of u_3 that is not on C.

Case 1: Assume u_4 is a $5s$-vertex.

Since $G - u$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3, 4)$-coloring φ of $G - u$. By Lemma 2.3 (ii), u_3 is 3-saturated and u_3 has a 4-saturated 6^+-neighbor, which must be u_2. In particular, $\varphi(u_3) = \varphi(u_4) = 3$ and $\varphi(u_2) = 4$. Yet, if u_2 is a $5p$-vertex or a $6p$-vertex, then we can recolor u_2 and u_3 with the color 3 and the color 4, respectively, and color u with the color 3 to obtain a $(3, 4)$-coloring of G, which is a contradiction. By symmetry, u_5 is also neither a $5p$-vertex nor a $6p$-vertex. Therefore, if both u_3 and u_4 are $5s$-vertices, then C has at most two of $5p$-, $5s$-, and $6p$-vertices.

Case 2. Assume u_4 is a $6s^+$-vertex. Now u_2 is a $5s$-vertex or $6p$-vertex, and u_5 is a $5p$-, $5s$-, or $6p$-vertex.

First assume u_2 is a $5s$-vertex. Since $G - u_1$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3, 4)$-coloring φ of $G - u_1$. By Lemma 2.3 (i), u_2 must be 3-saturated and u_5 must be a 4-saturated $6p$-vertex. This further implies that u_4 is 3-saturated. Note that u_2 must have a 4-saturated neighbor and three neighbors of color 3. Since $\varphi(u_4) = 3$, we know u_3 cannot be the 4-saturated neighbor of u_2, so $\varphi(u_3) = 3$. Now, since u_3 has neither five neighbors colored with the color 4 nor a 4-saturated neighbor, u_3 can be recolored with 4. Now, by recoloring u_3 with the color 4 and coloring u_1 with the color 3, we obtain a $(3, 4)$-coloring of G, which is a contradiction.

Now assume u_2 is either a $5p$-vertex or a $6p$-vertex. Since $G - u$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3, 4)$-coloring φ of $G - u$. By Lemma 2.3 (ii), u_3 is 3-saturated and u_3 has a 4-saturated 6^+-neighbor x. If $x = u_2$, then we can recolor u_2 with the color 3, and color u_3 and u with the color 4 and 3, respectively, to obtain a $(3, 4)$-coloring of G, which is a contradiction. Therefore $x = u_4$, which implies that $\varphi(u_4) = 4$ and $\varphi(u_3) = 3$. Since recoloring u_3 with the color 4 must not be possible, we know that all neighbors of u_2, except u_3, are colored with the color 4. In particular, $\varphi(u_1) = 4$. This further implies that u_5 is 3-saturated and non-recolorable. Now, u_4 must have four neighbors colored with the color 4, and at least one neighbor colored with the color 3 that is not on C. Hence, C is either F_{5c} or F_{5d}. □

{lem:7-face}

Lemma 2.7. *If F is a 7-face, then one of the following is true:*

- *F has at most six 2-, 5p-, 5s-, or 6p-vertices;*
- *F has at least two (adjacent) 5s-vertices;*
- *F is a special 7-face F_7 (see Figure 2).*

Proof. Note that two 2-vertices cannot be adjacent to each other by Lemma 2.1. Suppose to the contrary that F contains seven of 2-, $5p$-, $5s$-, and $6p$-vertices, and at most one $5s$-vertex. Without loss of generality, we may assume that one vertex u_1 is a $5s$-vertex, for otherwise, two $6p^-$-vertices would be adjacent to each other, which contradicts Lemma 2.3 (*ii*) and (*iii*). All other vertices of F are 2-vertices and $6p^-$-vertices.

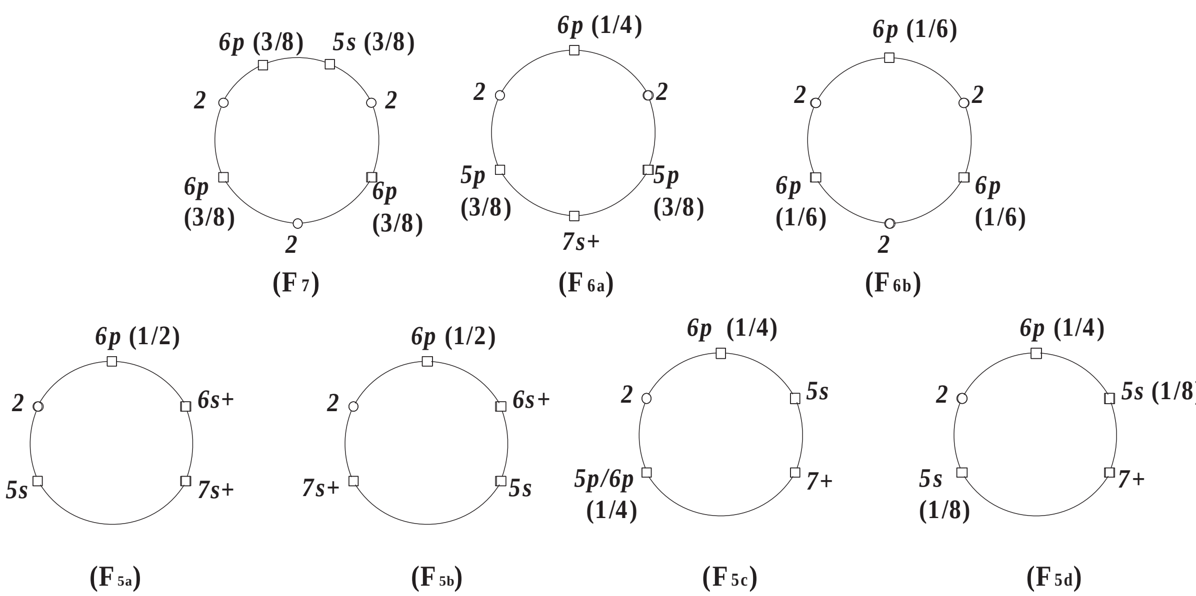
Without loss of generality, we may assume that u_2, u_4, u_6 are 2-vertices and u_3, u_5, u_7 are $6p^-$vertices. Since u_2 is a 2-vertex, by Lemma 2.3 (*i*), we know u_3 is a $6p$-vertex. Since a $5p$-vertex cannot have a $5s$-neighbor by Lemma 2.3 (*ii*), we now u_7 must be a $6p$-vertex. If u_5 is a $6p$-vertex, then F is a special face F_7.

The only remaining case is when u_5 is a $5p$-vertex and u_3, u_7 are $6p$-vertices. Since $G - u_4$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there is a $(3,4)$-coloring φ of $G - u_4$. By Lemma 2.3 (*i*), u_3 and u_5 is 3-saturated and 4-saturated, respectively. In particular, $\varphi(u_3) = \varphi(u_2) = 4$ and $\varphi(u_5) = \varphi(u_6) = 3$. This further implies that u_1 is 3-saturated and u_7 is 4-saturated. Now, recoloring u_2, u_1, u_7 with the color $3, 4, 3$, respectively, and coloring u_4 with the color 4 gives a $(3,4)$-coloring of G, which is a contradiction. □

3. Discharging

For each element $x \in V(G) \cup F(G)$, let $\mu(x)$ and $\mu^*(x)$ denote the *initial charge* and *final charge*, respectively, of x. Let $\mu(x) = d(x) - 4$, so by Euler's formula,

$$\sum_{x \in V(G) \cup F(G)} \mu(x) = -8.$$



cial-6-face}

Figure 2. Special 7-face, 6-faces, and 5-faces

Here are the discharging rules:

(R1) Let v be a 5^+-vertex. Then v gives $\frac{1}{2}$ to each adjacent 2-vertex; moreover,
   (a) If $d(v) \geq 8$, then v gives $\frac{1}{2}$ to each adjacent $5p$-, $5s$-, $6p$-vertex and incident heavy edge.
   (b) If $d(v) = 7$, then v first gives $\frac{1}{2}$ to each adjacent $5p$-vertex and $6p$-vertex, then distributes its remaining charge evenly to adjacent $5s$-vertices and incident heavy edges.

(c) If $d(v) \in \{5,6\}$, then v distributes its remaining charge evenly to its adjacent $5p$-vertices and incident heavy edges.

(R2) A heavy edge distributes its charge evenly to the two incident faces.

(R3) Let f be a 5^+-face. Then f gives $\frac{1}{2}$ to each incident 2-vertex; moreover,

(a) If $d(f) \geq 8$, then f gives $\frac{1}{2}$ to each incident $5p$-, $5s$-, and $6p$-vertex.

(b) Let $d(f) = 7$. If $f \neq F_7$, then f first gives $\frac{1}{2}$ to each incident $5p$-vertex and $6p$-vertex, then distributes its remaining charge evenly to each incident $5s$-vertex (if exists). If $f = F_7$, then f gives $\frac{3}{8}$ to each incident $5s$-vertex and $6p$-vertex.

(c) Let $d(f) = 6$. If $f \neq \{F_{6a}, F_{6b}\}$, then f first gives $\frac{1}{2}$ to each incident $5p$-vertex and $\frac{1}{4}$ to each incident $5s$-vertex and $6p$-vertex, then distributes its remaining charge evenly to each incident $5p$-, $5s$-, and $6p$-vertex (if exists). If $f = F_{6a}$, then f gives $\frac{3}{8}$ to each incident $5p$-vertex and $\frac{1}{4}$ to the incident $6p$-vertex. If $f = F_{6b}$, then f gives $\frac{1}{6}$ to each incident $6p$-vertex.

(d) Let $d(f) = 5$. If f is incident with two 2-vertices, then it distributes its charge evenly to each incident $5p$-, $5s$-, and $6p$-vertex (if exists). If f has at most one 2-vertex and $f \notin \{F_{5a}, F_{5b}, F_{5c}, F_{5d}\}$, then f first gives $\frac{1}{4}$ to each incident $5p$-, $5s$-, and $6p$-vertex, then it distributes its remaining charge evenly to each incident $5p$-, $5s$-, and $6p$-vertex (if exists). If $f \in \{F_{5a}, F_{5b}\}$, then it gives $\frac{1}{2}$ to the incident $6p$-vertex and its remaining charge to the $5s$-vertex. If $f \in \{F_{5c}, F_{5d}\}$, then it gives $\frac{1}{4}$ to each incident $5p$-vertex and $6p$-vertex, and its remaining charge evenly to incident $5s$-vertices.

{lem-face-ch

Lemma 3.1. *If f is a 5^+-face, then $\mu^*(f) \geq 0$.*

Proof. If f is a 5-face, then $\mu(f) = 1$ and f is incident with at most two 2-vertices. Clearly, $\mu^*(f) \geq 1 - 1 = 0$ by (R3d) and Lemma 2.6. If f is a 7-face, then $\mu(f) = 3$. By (R3b) and Lemma 2.7, $\mu^*(f) \geq 7 - 4 - \frac{1}{2} \cdot 6 = 0$. If f is a 8^+-face, then $\mu^*(f) \geq d(f) - 4 - \frac{1}{2}d(f) \geq 0$ by (R3a).

If f is a 6-face, then $\mu(f) = 2$ and f is incident with at most three 2-vertices by Lemma 2.1.

- f has at most one incident 2-vertex.
  By Lemma 2.4 (c) and (d), and (R3c), $\mu^*(f) \geq 2 - \max\{\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4, \frac{1}{4} \cdot 6\} = 0$
- f has two incident 2-vertices.
  By Lemma 2.4 (b), f has at most two incident $5p$-vertices. If f has no incident $5p$-vertex, then $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 4 = 0$. If f has one incident $5p$-vertex, then f has at most two of $5s$-vertices and $6p$-vertices by Lemma 2.4 (b), so $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 3 - \frac{1}{4} \cdot 2 = 0$. If f has two incident $5p$-vertices, then f is either a special face F_{6a} or has neither $5s$-vertices nor $6p$-vertices. Therefore, $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 2 - \frac{3}{8} \cdot 2 - \frac{1}{4} = 0$ or $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 4 = 0$.
- f has three incident 2-vertices.
  If f is incident with a 5-vertex, then the other two vertices on f are 7^+-vertices by Lemma 2.4 (a), so $\mu^*(f) \geq 2 - \max\{\frac{1}{2} \cdot 4, \frac{1}{2} \cdot 3 + \frac{1}{4}\} = 0$. If f is a special face F_{6b} in Figure 2, then $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 3 - \frac{1}{6} \cdot 3 = 0$. If f is not F_{6b}, then $\mu^*(f) \geq 2 - \frac{1}{2} \cdot 3 - \frac{1}{4} \cdot 2 = 0$.

□

Lemma 3.2. *If u is a $5p$-vertex, then $\mu^*(u) \geq 0$.*

Proof. By (R1), u gives out $4 \cdot \frac{1}{2} = 2$ to its adjacent 2-vertices. To show $\mu^*(u) \geq 0$, we need to prove that u receives at least 1 by the discharging rules. Let $N(u) = \{u_0, v_1, v_2, v_3, v_4\}$ where $d(u_0) > 2$

and $d(v_i) = 2$ for $i \in [4]$. For $i \in [4]$, let u_i be the neighbor of v_i that is not u. We assume that the five faces incident with u are A, B, C, D, E as shown in Figure 3.

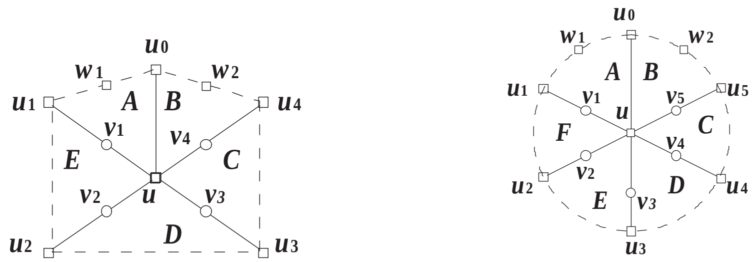


FIGURE 3. A $5p$-vertex incident with five 5^+-faces and a $6p$-vertex incident with six 5^+-faces. {figure-5p}

To get some idea regarding the degrees of the vertices on the five faces incident with u, we consider a $(3,4)$-coloring φ of $G - u$, which exists since the number of edges decreased and the number of 3^+-vertices did not increase. By Lemma 2.3 (ii), u_0 is a 4-saturated 6^+-vertex and the four 2-neighbors of u are colored with the color 3. Since u_0 is non-recolorable, if $d(u_0) \leq 8$, then u_0 has a 3-saturated neighbor and four neighbors of color 4. Furthermore, since no neighbor of u is recolorable, for $i \in [4]$, u_i is a 4-saturated 6^+-neighbor and if $d(u_i) \leq 8$, then u_i has a 3-saturated neighbor.

Case 1. u is incident with a special 6-face F_{6a}.

By the ordering of the degrees of the vertices on F_{6a}, the special 6-face must be either A or B. Without loss of generality, assume A is a special 6-face F_{6a} so that u_1 is a $6p$-vertex and u_0 is a $7s^+$-vertex. As both u_1 and u_2 are 4-saturated, and u_1 is adjacent to a 3-saturated vertex, we conclude u_1 cannot be adjacent to u_2. Otherwise, u_1 has two 3^+-neighbors, which implies u_1 is not a poor vertex. Hence, E is a 6^+-face. By (R3), u gets $\frac{1}{2}$ from E and $\frac{3}{8}$ from A, and by (R1), u gets $\frac{1}{2}$ from u_0. So u gets at least 1 in total, as desired.

Case 2. u is not incident with a special 6-face and either A or B is a non-special 6^+-face.

Note that by (R3), u receives at least $\frac{1}{2}$ from each of its incident 6^+-faces that are not special. So we may assume that u is incident with exactly one 6^+-face and four 5-faces. Without loss of generality, assume A is a 6^+-face and let $B = uu_0w_2u_4v_4$. Since a 3^+-neighbor u_3 of u_4 is 4-saturated, we know u_4 cannot be a $6p$-vertex. Therefore, B is not a special 5-face.

(1) We may assume u_0 is a 6-vertex. For otherwise, u also gets $\frac{1}{2}$ from u_0 by (R1), thus u gets at least 1 in total.
(2) We may assume w_2 is not a 2-vertex. For otherwise, as $\varphi(u_0) = \varphi(u_4) = 4$, this implies that $\varphi(w_2) = 4$, but now w_2 can be recolored, which is a contradiction.
(3) We may assume w_2 is a $5s$-vertex. For otherwise, none of u_0, w_2, u_4 is a $5p$-, $5s$-, or $6p$-vertex, so u receives at least $\frac{1}{2}$ from B by (R3d), thus u get at least 1 in total.
(4) We may assume each of u_3 and u_2 is either a $6r^+$-vertex or a 9^+-vertex, and u_1 is either a $6s^+$-vertex or 9^+-vertex. For $z \in \{u_3, u_2, u_1\}$, observe that each z must have either a 3-saturated neighbor or four vertices colored with the color 3.
(5) We may assume u_4 is either a 8^+-vertex or a $7r$-vertex. It must be that $\varphi(w_2) = 3$, for otherwise, we can recolor w_2 with the color 4 and color u with 3 to obtain a $(3,4)$-coloring of G, which is a contradiction. Now u_4 must have a 3-saturated neighbor that is not w_2,

for otherwise, we could recolor u_0, w_2, u_4 with $3, 4, 3$, respectively. This implies that u_4 has at least three 3^+-neighbors, so must u_4 is either a 8^+-vertex or a $7r$-vertex.

Now, u_4u_3, u_3u_2, u_2u_1 are all heavy edges. For $i \in [4]$, if $d(u_i) \leq 8$, then it is not a $5p$-, $5s$-, and $6p$-vertex. Thus, by (R1), the heavy edges u_4u_3, u_3u_2, u_2u_1 get at least $\frac{1}{3} + \frac{1}{6}$, $\frac{1}{6} \cdot 2$, $\frac{1}{6} \cdot 2$ from u_4, u_3, u_2, respectively. By (R2) and (R3), u receives at least $\frac{1}{2}(\frac{1}{3} + \frac{1}{6} + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 2) > \frac{1}{2}$ from faces C, D, E, and thus a total of 1, as desired.

Case 3. u is not incident with a speical 6-face and both A and B are 5-faces.

Let $A = uu_0w_1u_1v_1$ and $B = uu_0w_2u_4v_4$.

(1) If k is the number of vertices in $\{w_1, w_2\}$ that is either a 2-vertex or a $5s$-vertex, then $d(u_0) \geq 6 + k$. This is because if w_i is either a 2-vertex or a $5s$-vertex, then $\varphi(w_i) = 3$, otherwise we can recolor w_i with 3 and color u with 4 to obtain a $(3, 4)$-coloring of G, which is a contradiciton. The lower bound on $d(u_0)$ follows since u_0 is 4-saturated and cannot be recolored with the color 3.

(2) We may assume C, D, E are 5-faces. For otherwise, u gets at least $\frac{1}{2}$ from an incident 6^+-face by (R3). Now, if $d(u_0) \geq 7$, then u gets another $\frac{1}{2}$ from u by (R1), for a total of 1. If $d(u_0) = 6$, then neither w_1 nor w_2 is a 2-vertex, and both u_1 and u_4 are $6s^+$-vertices. Since neither A nor B is a special face, by (R3), u gets at least $\frac{1}{4} \cdot 2$ from A and B, for a total of 1.

(3) We observe each of u_3 and u_2 is either a 9^+-vertex or a $6r^+$-vertex. This follows since each of u_3 and u_2 has two 4-saturated neighbors and is not recolorable with the color 3.

(4) Assume $d(u_1), d(u_4) \leq 8$. For $i \in [2]$, u_{3i-2} is a $7s^+$-vertex if $d(w_i) = 2$ and is a $6s^+$-vertex if $d(w_i) \geq 3$.

Now, u_1u_2, u_2u_3, u_3u_4 are all heavy edges. By (R1), u_2 sends at least $\min\left\{\frac{6-4-3\cdot\frac{1}{2}}{3}, \frac{7-4-5\cdot\frac{1}{2}}{2}, \frac{1}{2}\right\} = \frac{1}{6}$ to each of u_1u_2 and u_2u_3, and likewise, u_3 sends at least $\frac{1}{6}$ to each of u_2u_3 and u_3u_4.

- Assume both w_1 and w_2 are 2-vertices. Now, u_0 is a 8^+-vertex and gives $\frac{1}{2}$ to u by (R1a). Also, u_1 and u_4 is a $7s^+$-vertex and gives at least $\frac{7-4-5\cdot\frac{1}{2}}{2} = \frac{1}{4}$ to the heavy edge u_1u_2 and u_3u_4, respectively, by (R1). By (R2) and (R3), C, D, E give at least $\frac{1}{2}(\frac{1}{6} \cdot 4 + \frac{1}{4} \cdot 2) > \frac{1}{2}$ to u.
- Without loss of generality, assume w_1 is a 2-vertex and w_2 is a 3^+-vertex. Now, u_0 is a 7^+-vertex and gives $\frac{1}{2}$ to u by (R1), and the face B gives at least $\frac{1}{4}$ to u by (R3). Also, u_1 is a $7s^+$-vertex and gives $\frac{7-4-5\cdot\frac{1}{2}}{2} = \frac{1}{4}$ to the heavy edge u_1u_2. Then u gets at least $\frac{1}{2} + \frac{1}{4}$ from u_0 and B, and at least $\frac{1}{2}(\frac{1}{6} \cdot 4 + \frac{1}{4}) > \frac{1}{4}$ from C, D, E by (R3).
- Finally, assume that none of w_1, w_2 is a 2-vertex. By (R3d), each of A, B gives at least $\frac{1}{4}$ to u, and moreover, at least $\frac{1}{2}$ to u if neither w_1 nor w_2 is a $5s$-vertex. Now if either w_1 or w_2 is a $5s$-vertex, then u_0 is a 7^+-vertex, and thus u_0 gives $\frac{1}{2}$ to u so u gets a total of $\frac{1}{4} \cdot 2 + \frac{1}{2} \geq 1$.

Hence, u always gets at least 1, as desired. □

Lemma 3.3. *If u is a $6p$-vertex, then $\mu^*(u) \geq 0$.*

Proof. The initial charge of u is 2, and by (R1c), u gives out $\frac{1}{2} \cdot 5$ to its 2-neighbors. To show $\mu^*(u) \geq 0$, we need to prove that u receives $\frac{1}{2}$ by the discahrging rules.

Let $N(u) = \{u_0, v_i : i \in [5]\}$ where $d(u_0) > 2$ and $d(v_i) = 2$ for $i \in [5]$. For $i \in [5]$, let u_i be the neighbor of v_i that is not u. We assume that the six faces incident with u are A, B, C, D, E, F as shown in Figure 3.

Since $G - u$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there exists a $(3,4)$-coloring φ of $G - u$. By Lemma 2.3, either $\varphi(v_i) = 4$ for $i \in [5]$ and u_0 is 3-saturated and non-recolorable, or at least four of v_i's are colored with 3 and u_0 is 4-saturated and non-recolorable. In the former case, u_i with $i \in [5]$ are 3-saturated and non-recolorable, and in the latter case, at least four of the u_i's are 4-saturated and non-recolorable.

{i-u0}

(1) $d(u_0) \leq 6$
By (R1), u gets $\frac{1}{2}$ from u_0 if $d(u_0) \geq 7$.

(2) u is not incident with a special face F_{6b}.
If u is incident with F_{6b}, then by Lemma 2.5, u is also incident with two faces where each face is not a non-special 5-face with two 2-vertices. By (R3), each face that is not a 5-face with two 2-vertices sends at least $\frac{1}{4}$ to u, plus F_{6b} sends $\frac{1}{6}$ to u. Thus, u gets a total of at least $\frac{1}{2} + \frac{1}{6}$.

(3) We may assume A is a 5-face with two 2-vertices.
By (R3), each face that is not a 5-face with two 2-vertices gives at least $\frac{1}{4}$ to u, so either A or B must be a 5-face with two 2-vertices, which we may assume to be A.

(4) B is not a 5-face with two 2-vertices, and C, D, E, F are 5-faces with two 2-vertices.
Suppose B is a 5-face with two 2-vertices, so that both w_1 and w_2 are 2-vertices. If u_0 is 3-saturated, then both u_1 and u_5 are 3-saturated. So w_1, w_2 are colored or can be recolored with 4. This implies that u_0 is a 7^+-vertex, which contradicts (1). If u_0 is 4-saturated, then either u_1 or u_5 is 4-saturated. Without loss of generality assume u_1 is 4-saturated, so either $\varphi(w_1) = 3$ or w_1 can be recolored with 3. This implies that u_0 is a 7^+-vertex, which contradicts (1).

Now u receives at least $\frac{1}{4}$ from B by (R3). This implies that C, D, E, F are 5-faces with two 2-vertices, for otherwise, u receives another $\frac{1}{4}$ to get a total of at least $\frac{1}{2}$.

(5) B is not a special face F_7.
Without loss of generality, assume B is a special face F_7, which sends $\frac{3}{8}$ to u. This implies that u_0 is a $5s$-vertex, which futher implies that u_0 is 3-saturated and u_i is 3-saturated for each $i \in [5]$. By Lemma 2.3, we know $\varphi(w_1) = 3$. If w_1 is a 2-vertex, then recolor w_1 and v_1 with the color 4 and 3, respectively, and color u with the color 4 to obtain a $(3,4)$-coloring of G, which is a contradiction. If w_1 is not a 2-vertex, then u gets at least $\frac{1}{4}$ from A by (R3), so u gets a total of $\frac{1}{4} + \frac{3}{8} > \frac{1}{2}$.

{i-5Fab}

(6) B is a 6^--face. Moreover, if B is a 5-face, then it can be neither F_{5a} nor F_{5b}.
Otherwise, u receives at least $\frac{1}{2}$ by (R3a), (R3b), (R3d).

{i-6face}

(7) B must be a 6-face.
Suppose otherwise. From above, assume B is a 5-face with at most one 2-vertex. Note that B must have exactly one 2-vertex since v_5 is a 2-vertex. By (R3), B gives u at least $\frac{1}{2}$ if u is the only $5p$-, $5s$-, or $6p$-vertex on B. So consider the case when B is a 5-face with one 2-vertex v_5 and at least two $5p$-, $5s$-, or $6p$-vertices. Note that none of u_0, w_2, u_5 can be a $6p^-$-vertex.

Assume u_0 is 3-saturated. Then $\varphi(v_i) = 4$ and u_i is 3-saturated for $i \in [5]$. The 2-vertex w_1 is colored or can be recolored with 4. Therefore u_0 is a $6s$-vertex. Thus, either u_5 or w_2 is a $5s$-vertex. Since B is not F_{5a} or F_{5b} by (6), when one of u_5 and w_2 is a $5s$-vertex, the other one is a 6^--vertex. Now if w_2 is a $5s$-vertex, then w_2 is colored or can be recolored with 4 without making w_2 4-saturated, so u_0 must have another 4-saturated neighbor. Thus, $d(u_0) \geq 7$, which contradicts (1). If u_5 is a $5s$-vertex, then w_2 must be the 4-saturated neighbor of u_5 and u_0. Thus, we can recolor u_0, u_5 with 4 and w_2 with 3, and color u with 3 to obtain a $(3,4)$-coloring of G, which is a contradiction.

Assume u_0 is 4-saturated. Then u_0 is $6s^+$-vertex. Now, since $d(u_0) \leq 6$, the 2-vertex w_1 cannot be colored or recolored with 3. This implies that $\varphi(w_1) = 4$ and $\varphi(u_1) = 3$, and moreover, $\varphi(v_1) = 4$. Furthermore, for $i \in [5] - \{1\}$, $\varphi(v_i) = 3$ and u_i is 4-saturated. Since u_5 is 4-saturated, it is $6s^+$-vertex. So w_2 is a $5s$-vertex, and $\varphi(w_2) = 3$ or w_2 can be recolored with 3. Again, since B is neither F_{5a} nor F_{5b}, we know $d(u_5) \leq 6$. Then w_2 is the only 3-saturated neighbor of u_0 and u_5. So by recoloring u_0, w_2, u_5 with $3, 4, 3$, and coloring u with 4, we obtain a $(3,4)$-coloring of G, which is a contradiction. {i-2vx}

(8) If $B = uu_0w_2w_2'u_5v_5$, then either w_2 or w_2' is a 2-vertex.
For otherwise, B contains exactly one 2-vertex v_5. Moreover, the only $6p^-$-vertex B contains is u. We may assume that B contains at least three $5s$-vertices, for otherwise u gets at least $\frac{6-4-0.5}{3} = 0.5$ from B by (R3c). Since no $5s$-vertex can be adjacent to two $5s$-vertex, by Lemma 2.3, we know either w_2 or w_2' is not a $5s$-vertex, and both u_0 and u_5 are $5s$-vertices. Now, both u_0 and u_5 cannot be 4-saturated, thus they are both 3-saturated. Moreover, $\varphi(v_i) = 4$ and $\varphi(u_i) = 3$ for $i \in [5]$. Now we can recolor w_1 with 4 and color u with 3 to obtain a $(3,4)$-coloring of G, which a contradiction. {i-two5s}

(9) B contains at least two $5s$-vertices.
For otherwise, B contains at most one $5s$-vertex. Note that the only $6p^-$-vertex B contains is u. Then by (R3c), B gives at least $\frac{6-4-0.5\cdot 2}{2} = \frac{1}{2}$ to u.

(10) u_0 must be 4-saturated.
For otherwise, u_0 is 3-saturated. It follows that $\varphi(v_i) = 4$ and u_i is 3-saturated for $i \in [5]$. So $\varphi(w_1) = 4$, and thus u_0 is a 6-vertex with a 4-saturated neighbor. Now if w_2 is a 2-vertex, then it must be that $\varphi(w_2) = 3$, otherwise we can recolor u_3 with 4 and color u with 3 to obtain a $(3,4)$-coloring of G, which is a contradiction. It follows that $\varphi(w_2') = 4$ and w_2' cannot be recolored, thus $d(w_2') \geq 6$, and therefore B contains at most one $5s$-vertex, which contradicts (9).

Thus w_2' is a 2-vertex. It follows that both w_2 and u_5 are $5s$-vertices. Since a 3-saturated vertex u_5 must have a 4-saturated neighbor, it must be that $\varphi(w_2') = 3$. We can recolor w_2' with the color 4 since w_2 is a 5-vertex with a neighbor u_0 colored with 3. Now, recolor v_5 with 3 and color u with 4 to obtain a $(3,4)$-coloring of G, which is a contradiction.

Since u_0 is 4-saturated and non-recolorable, by (1) we know u_0 is a 6-vertex. Moreover, $\varphi(w_1) = 4$, which further implies that u_1 is 3-saturated and non-recolorable. Therefore $\varphi(v_1) = 4$ and for $i \in [5] - \{1\}$, $\varphi(v_i) = 3$. In particular, $\varphi(v_5) = 3$ and u_5 is 4-saturated. However, by (7), (8), and (9), we know u_5 must be a $5s$-vertex, so we can recolor u_5 and v_5 with 3 and 4, and color u with 3 to obtain a $(3,4)$-coloring of G, which is a contradiction. □

Lemma 3.4. *If u is a $5s$-vertex, then $\mu^*(u) \geq 0$.*

Proof. The initial charge of u is 1, and by (R1c), u gives out $\frac{1}{2} \cdot 3$ to its 2-neighbors. To show $\mu^*(u) \geq 0$, we need to prove that u receives $\frac{1}{2}$ by the discharging rules.

Let $N(u) = \{x, y, v_1, v_2, v_3\}$ with $d(x), d(y) > 2$ and $d(v_i) = 2$ and let u_i be the other neighbor of v_i for $i \in [3]$. Depending on whether x, y, u are on the same face or not, we could have two different embeddings around u (see Figure 4).

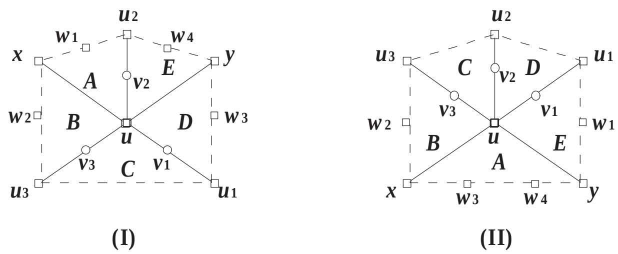


FIGURE 4. Two possible embeddings containing $5s$-vertex u with five 5-faces. {figure-5s}

Since $G - v_2$ is a graph with fewer edges than G and the number of 3^+-vertices did not increase, there exists a $(3, 4)$-coloring φ of $G - v_2$. By Lemma 2.3, u is 3-saturated and u_2 is 4-saturated, and both are non-recolorable. Without loss of generality, we may assume that x is 4-saturated and $\varphi(y) = \varphi(v_1) = \varphi(v_3) = 3$. Also, u_1 and u_3 are 4-saturated and non-recolorable.

We may assume $d(x), d(y) \leq 7$, for otherwise, u gets at least $\frac{1}{2}$ by (R1a).

Case 1. x, y, u are not on the same face (see Figure 4 (I) for an illustration).

(1) u is not incident with a special 5-face F_{5b} or F_{5c}.
This is because u is adjacent to a 2-vertex on each incident face.

(2) None of B, D is a special 5-face F_{5a} or F_{5d}. It follows that none of B, D are special 5-faces.
By symmetry, let B be a special 5-face F_{5a} or F_{5d}. Then x is a $7s^+$-vertex, w_2 is a $5s^+$-vertex, and u_3 is a $6p$-vertex. So $u_1u_3 \notin E(G)$, and C must be a 6^+-face. By (R1), u receives at least $\frac{1}{4}$ from x, and by (R3), u receives at least $\frac{1}{4}$ from C.

(3) We may assume at least one of B, D is a 5-face with two 2-vertices. Furthermore, we may assume that $d(w_3) = 2$.
If none of B, D are 5-faces with two 2-vertices, then they are 6^+-faces or 5-faces with at most one 2-vertex, so u gets at least $\frac{1}{4}$ from each of them by (R3).
If $d(w_3) > 2$, then D gives at least $\frac{1}{4}$ to u by (R3), and by what we just proved, B must be a 5-face with two 2-vertices. So $d(w_2) = 2$ and w_2 can be recolored with 3. Then x must be a 7^+-vertex. So u gets at least $\frac{1}{4}$ from x, thus gets at least $\frac{1}{2}$.

(4) $d(w_1) > 2$ and $d(w_2) = 2$, and moreover, A is a special 5-face.
First of all, at most one of w_1 and w_2 is a 2-vertex. Suppose otherwise. Then w_2 can be recolored with 3 and thus x must be a $7s^+$-vertex. So u receives at least $\frac{1}{4}$ from x. On the other hand, if A is not a 5-face, then u gets another $\frac{1}{4}$ from A. So let A be a 5-face. Then w_1 can be recolored with 3, since $\varphi(u_2) = \varphi(x) = 4$. Now that x is 4-saturated and non-recolorable, x must be adjacent to other neighbors than w_1, w_2, u of color 3, and four neighbors of color 4, so it is a 8^+-vertex. By (R1), u gets at least $\frac{1}{2}$ from x.
Now assume that $d(w_1) = 2$ and $d(w_2) > 2$. Then w_1 can be recolored with 3, so x is a 7^+-vertex. By (R1), u gets at least $\frac{1}{4}$ from u. If B is not Z, then by (R3), u gets at least $\frac{1}{4} \cdot 2$ from B well. If B is Z, then u_3 is a $6p$-vertex, and thus $u_3u_1 \notin E(G)$ and C is a 6^+-face, so u gets at least $\frac{1}{4}$ from C. In either case, u gets at least $\frac{1}{2}$. So $d(w_1) > 2$ and $d(w_2) = 2$.

Suppose that A is not a special 5-face. Then by (R3), u gets at least $\frac{1}{4}$ from A and we still have $d(x) \geq 7$, so by (R1) u gets at least $\frac{1}{4}$ from x as well.

(5) A must be F_{5a}.
For otherwise, by (1) and (4), A must be F_{5d}. Then $d(x) \geq 7$, u_2 is a $6p$-vertex and w_1 is a $5s$-vertex. So w_1 is the only 3-saturated neighbor of the $6p$-vertex u_2. Note that $d(x) \leq 7$, for otherwise, u gets $\frac{1}{2}$ from x by (R1). Now that w_2 can be recolored with 3 and x is 4-saturated, we can recolor x, w_1, u_2, u with $3, 4, 3, 4$, respectively, then color v_2 with 4, a contradiction.

(6) We claim that x must be a $7s$-vertex and w_1 must be a $6s$-vertex.
For otherwise, by (R1), if x is a $7r^+$-vertex, then x gives at least $\frac{7-4-4\cdot\frac{1}{2}}{3} = \frac{1}{3}$ to u and the heavy edge xw_1. By (R2) and (R3d), u gets a total $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Likewise, if w_1 is a $6r^+$-vertex or a 7^+-vertex, then w_1 gives at least $\frac{6-4-3\cdot 0.5}{2} = \frac{1}{4}$ to the heavy edge w_1x; note that x gives at least $\frac{1}{4}$ to u and the heavy edge xw_1, so u gets at least $\frac{1}{2}$ from A and x.

Then w_1 is the only 3-saturated neighbor of x and u_2 is the only 4-saturated neighbor of w_1. Note that u_2 is a $6p$-vertex by the definition of F_{5a}. Now we can recolor x, w_1, u_1 with $3, 4, 3$, respectively, and color u, v_2 with $4, 3$, respectively, a contradiction.

Case 2. x, y, u are in the same face, denoted by A (see for example Figure 4 (II)).

(1) u must be incident with a special 5-face F_{5a}, F_{5b}, F_{5c} or F_{5d}.
Assume that u has none of the special 5-faces. By (R3), each 6^+-face or 5-face with at most one 2-vertex gives at least $\frac{1}{4}$ to u, in particular, A gives $\frac{1}{4}$ to u. So all other faces are 5-faces with two 2-vertices. This implies that $d(w_1) = d(w_2) = 2$.
Recall that x and u_3 are 4-saturated. Then w_2 is colored or can be recolored with 3. Note that $d(x) \leq 6$, for otherwise, u gets $\frac{1}{4}$ from x. So u, w_2 are the only neighbors of x of color 3. Now recolor x with 3 and u with 4, and we can color v_2 with 3, a contradiction.

(2) none of B, E is a special 5-face.
If B or E is a special 5-face, then they only could be in $\{F_{5a}, F_{5d}\}$. By symmetry, assume that B is a special 5-face. Then u_3, w_2, x are $6p$-, $5s^+$- and $7s^+$-vertices, respectively. Since u_3 is a $6p$-vertex, $u_3u_2 \notin E(G)$, so C is a 6^+-face, thus u gets at least $\frac{1}{4}$ from C by (R3c). So u gets at least $\frac{1}{2}$ since u gets at least $\frac{1}{4}$ from x by (R1) at well.

(3) A cannot be a special 5-face.
Clearly, A cannot be a special 5-face F_{5a}. So we may assume that A is a special 5-face in $\{F_{5b}, F_{5c}, F_{5d}\}$. So one of x and y is 7^+-vertex, and by (R1), u receives at least $\frac{1}{4}$ from it. We may assume that B, E are are 5-faces with two 2-vertices (for otherwise, u receives at least $\frac{1}{4}$ from them). Then $d(w_1) = d(w_2) = 2$.
Now that w_2 is colored or can be recolored with 3. Since u_3 is 4-saturated and non-recolorable, u_3 must be a 7^+-vertex. Note that u_2 must be a $6r^+$-vertex and u_1 be a $6s^+$-vertex. By (R1), u_2 gives at least $\frac{6-4-3\cdot 0.5}{3} = \frac{1}{6}$ to each of the heavy edges u_2u_3 and u_2u_1, and u_3 gives at least $\frac{7-4-5\cdot 0.5}{2} = \frac{1}{4}$ to the heavy edge u_2u_3. So by (R2) and (R3), u gets at least $\frac{1}{2}(\frac{1}{4} + \frac{1}{6} \cdot 2) > \frac{1}{4}$ from C and D. So u gets at least $\frac{1}{2}$.

Clearly, C, D cannot be special 5-faces, so we reach a contradiction. □

Lemma 3.5. *Every vertex $u \in V(G)$ has $\mu^*(u) \geq 0$.*

Proof. We consider the cases according to the degree of u. Clearly, $\mu^*(u) = 2 - 4 + 4 \cdot \frac{1}{2} = 0$ if $d(u) = 2$ by the rules. If $d(u) \geq 8$, then $\mu^*(u) \geq d(u) - 4 - d(u) \cdot \frac{1}{2} \geq 0$. For $d(u) \in \{5, 6, 7\}$, the lemmas have shown that $\mu^*(u) \geq 0$. Note that $d(u) \neq 3$ and 4-vertices have initial and final charges $4 - 4 = 0$. □

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