DP-4-COLORABILITY OF PLANAR GRAPHS WITHOUT GIVEN TWO ADJACENT CYCLES

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Abstract. DP-coloring (also known as correspondence coloring) is a generalization of list coloring introduced recently by Dvořák and Postle (2017). Kim and Ozeki proved that planar graphs without k-cycles where $k = 3, 4, 5$, or 6 are DP-4-colorable. In this paper, we prove that every planar graph G without k-cycles adjacent to triangles is DP-4-colorable for $k = 5, 6$, which implies that every planar graph G without k-cycles adjacent to triangles is 4-choosable for $k = 5, 6$. This extends the result of Kim and Ozeki on 3-, 5-, and 6-cycles.

1. Introduction

Coloring is one of the main topics in graph theory. A *proper k-coloring* of G is a mapping $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$, where $[k] = \{1, 2, \ldots, k\}$. The smallest k such that G has a k-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. List coloring was introduced by Vizing [19], and independently Erdős, Rubin, and Taylor [7]. A *list assignment* of a graph $G = (V, E)$ is a function L that assigns to each vertex $v \in V$ a list $L(v)$ of colors. An *L-coloring* of G is a function $\lambda : V \to \cup_{v\in V} L(v)$ such that $\lambda(v) \in L(v)$ for every $v \in V$ and $\lambda(u) \neq \lambda(v)$ whenever $uv \in E$. A graph G is *k-choosable* if G has an L-coloring for every assignment L with $|L(v)| \geq k$ for each $v \in V(G)$. The *choice number*, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable.

The techniques used in ordinary colorings may not be applicable for list coloring problems. For example, identifications of vertices are quite common in ordinary colorings, but not feasible in list coloring, since different vertices may have different lists, it is not possible for one to use identification of vertices. Motivated by this, Dvořák and Postle [6] introduced correspondence coloring (or DP-coloring) as a generalization of list coloring. The following equivalent definition is given by Bernsheteyn, Kostochka and Pron [3].

Definition 1.1. ([3]) *Let G be a graph. A cover of G is a pair (L, H), where $L : V(G) \to \mathbb{N}\times V(G)$ and H is a graph with vertex set $\cup_{v\in V(G)} L(v)$ satisfying the following conditions:*

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- *$H[L(v)]$ is a complete graph for each $v \in V(G)$.*
- *For each $uv \in E(G)$ the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).*
- *For every two distinct $u, v \in V(G)$ with $uv \notin E(G)$ no edges of H connect $L(u)$ and $L(v)$.*

Definition 1.2. *([3]) Suppose that G is a graph and (L, H) is a cover of G. An (L, H)-coloring of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. The graph G is said to be (L, H)-colorable if G admits an (L, H)-coloring.*

Definition 1.3. *([3]) Let G be a graph and let $f : V(G) \to \mathbb{Z}^+$ be an assignment of nonnegative integers to the vertices of G. The graph G is DP-f-colorable if G is (L, H)-colorable whenever (L, H) is a cover of G and $|L(v)| \geq f(v)$ for all $v \in V(G)$.*

The *DP-chromatic number* $\chi_{DP(G)}$ is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| = k$ for all $v \in V(G)$.

If for each $uv \in E(G)$, every edge in the matching between $L(u)$ and $L(v)$ has endpoints (i, u) and (i, v) with $i \in \mathbb{N}$, then an (L, H)-coloring is the usual list coloring. So list coloring is a special case of DP-coloring. In particular, $\chi_{DP}(G) \geq \chi_l(G)$ for each graph G.

Dvořák and Postle [6] mentioned that $\chi_{DP}(G) \leq 5$ if G is a planar graph, and $\chi_{DP}(G) \leq 3$ if G is a planar graph with girth at least 5. Also, Dvořák and Postle [6] observed that $\chi_{DP}(G) \leq k+1$ if G is k-degenerate. On the other hand, DP-coloring and list coloring are strikingly different. For instance, Bernshteyn [2] showed that the DP-chromatic number of every graph with average degree d is $\Omega(d/\log d)$, while Alon[1] proved that $\chi_l(G) = \Omega(\log d)$ and the bound is sharp.

The Four Color Theorem says that every planar graph is 4-colorable. Thomassen [18] showed that every planar graph is 5-choosable. Voigt [20] found a non-4-choosable planar graph. Moreover, Gutner [10] showed that it is NP-hard to determine whether a planar graph is 4-choosable. Thus, finding sufficient conditions for planar graphs to be 4-choosable is an interesting problem. It is known that a planar graph is 4-choosable if it has no i-cycle, where $i \in \{4, 5, 6, 7\}$, (see [8, 9, 14, 21]). For more results, the readers can see [4, 5, 11, 16, 17, 21, 23].

Some results of list coloring can be generalized to those of DP-coloring. Kim and Ozeki [12] proved that planar graphs without k-cycles where $k \in \{3, 4, 5, 6\}$ are DP-4-colorable. Chen, Chen and Wang [5] proved that every planar graph without 4-cycles adjacent to 3-cycles is 4-choosable which is generalized to DP-4-colorable by Kim and Yu [13].

In this paper, we prove the following two results.

Theorem 1.1. *Every planar graph G without 5-cycles adjacent to triangles has the minimum degree at most 3, and thus is DP-4-colorable.*

Theorem 1.2. *Every planar graph G without 6-cycles adjacent to triangles is DP-4-colorable.*

Theorems 1.1 and 1.2 extend the result of Kim and Ozeki on 3-, 5-, and 6-cycles.

The following are some notation used in the paper. Let F be the set of faces of G. A k-vertex (k^+-vertex, k^--vertex, respectively) is a vertex of degree k (at least k, at most k, respectively). The same notation will be applied to faces and cycles. A $(d_1, d_2, \ldots, d_k)$-face f is a face of degree k where all vertices on f have degree $d_1, d_2, \ldots, d_k$ in an arbitrary order. A $(d_1, d_2, \ldots, d_k)$-vertex

v is a vertex incident to exactly k faces where all faces incident to v have degree $d_1, d_2, \ldots, d_k$ in an arbitrary order, respectively. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside a cycle C, respectively. The cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$.

We use a discharging argument to prove Theorems 1.1 and 1.2. The proofs of Theorems 1.1 and 1.2 are given in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1

Let G be counterexample with minimum number of edges. Then the minimum degree of G is at least 4. Since G contains no 5-cycles adjacent to triangles, the following properties about G are straightforward.

Lemma 2.1. *(a) If a 4-vertex v is incident to a 4- or 5-face, then v is incident to at most one 3-face;*
(b) None of 4- or 5-faces is adjacent to a 3-face;
(c) Each 4^+-vertex v is incident to at most $(d(v) - 2)$ 3-faces.

We are now ready to complete the proof of Theorem 1.1 by way of a discharging procedure. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = 2d(v) - 6$, and each face f has an initial charge of $\mu(f) = d(f) - 6$. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = -12$.

Let $\mu^*(x)$ be the charge of $x \in V \cup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$.
The discharging rule:

(R1) Each 4^+-vertex gives 1 to each incident 3-face, then gives the remaining charge equally to each incident 4- or 5-face.

From the rule, we have the following useful observation.

Lemma 2.2. *Each 4- or 5-face gets at least $\frac{1}{2}$ from each incident 4^+-vertex.*

Proof. Suppose that v is a 4^+-vertex on a 4- or 5-face f. If $d(v) = 4$, then v is incident to at most one 3-face by Lemma 2.1(a). Furthermore, If v is incident to exactly one 3-face, then v has no incident 4- or 5-faces other than f by Lemma 2.1(b). By (R1) v gives at least $\frac{1}{2}$ to f. If $d(v) \geq 5$, then v has at most $(d(v) - 3)$ incident 3-faces. By (R1) f gets $\frac{2d-6-k}{d-k} \geq \frac{2}{3}$ from v, where $k \leq d(v) - 3$ denotes the number of incident 3-faces of v. $\square$

Proof of Theorem 1.1

It suffices to check that each $x \in V(G) \cup F(G)$ has nonnegative final charge.

First, we show that each vertex v in G has nonnegative final charge. By the assumption, $d(v) \geq 4$. By the discharging rule, we just need to confirm that after v gives charge to 3-faces, it has nonnegative charge. By Lemma 2.1(c), v is incident to at most $(d(v) - 2)$ 3-faces. So $\mu^*(v) \geq 2d(v) - 6 - (d(v) - 2) = d(v) - 4 \geq 0$.

Now, we show that each face f in G has nonnegative final charge. Observe that no rules are applied to 6^+-faces, therefore all such faces have a nonnegative final charge. Thus, we may assume that f is a k-face with $k \in \{3,4,5\}$. If f is a 3-face, then by (R1) each 4^+-vertex on f gives 1 to f. Thus $\mu^*(f) = -3 + 1 \times 3 = 0$. If f is a 4- or 5-face, then each 4^+-vertex on f gives at least $\frac{1}{2}$ to f by Lemma 2.2. Then $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} \times 4 \geq 0$.

This completes the proof.

3. Proof of Theorem 1.2

First, we introduce some definitions and notations that are used in the proof.

A graph $C(n_1, n_2, \ldots, n_k)$ is a plane graph obtained from an $(n_1 + n_2 + \cdots + n_k - 2k + 2)$-cycle with $k-1$ chords such that consecutive internal faces have length $n_1, n_2, \ldots, n_k$ in this order. Note that each $C(n_1, n_2, \ldots, n_k)$ is not necessary unique. For example, there are two non-isomorphic subgraphs that are $C(3,4,3)$, but exactly one isomorphic subgraph that is $C(3,3,3)$.

Let v be a vertex on a 3-face f. We call v a *good* vertex of f if f is not in $C(3,3,3)$. In a $C(3,3,3)$ that contains no vertices on C_0, we call the vertices in exactly one, two, or three 3-faces in the $C(3,3,3)$ *bad, worse, worst* vertices of the $C(3,3,3)$, respectively. So, there are two bad one, two worse ones, and exactly one worst one in any given $C(3,3,3)$. A *wheel graph* is a graph formed by connecting a single vertex (*hub*) to all vertices (*external vertices*) of a cycle. We use W_n to denote a wheel graph with n vertices. For our purpose, we regard each external vertex in W_5 as a worse vertex of a 3-face in some $C(3,3,3)$.

We prove the following statement, which is stronger than Theorem 1.2.

Theorem 3.1. *Let G be a planar graph without 6-cycles adjacent to triangles. Then any precoloring of a 3-cycle can be extended to a DP-4-coloring of G.*

Proof of Theorem 1.2 with Theorem 3.1: We may assume that G contains a 3-cycle C, for otherwise, G is DP-4-colorable by [12]. By Theorem 3.1, any precoloring of C can be extended to G, so G is also DP-4-colorable.

Let (G, C_0) be a minimal counterexample to Theorem 3.1, where C_0 is a 3-cycle in G that is precolored. If C_0 is a separating cycle, then any precoloring of C_0 can be extend to $int(C_0)$ and $ext(C_0)$, respectively. Then we get a DP-4-coloring of G, a contradiction. So we may assume that C_0 is the boundary of the outer face D of G in the rest of this paper.

Lemma 3.2. *Each vertex in $int(C_0)$ has degree at least four.*

Proof. Suppose otherwise that there exists a 3^--vertex $v \in int(C_0)$. Let $G' = G - v$. For each $w \in V(G')$, let $L'(w) = L(w)$ and let $H' = H - L(v)$. By the minimality of (G, C_0), (G', C_0) has an (L', H')-coloring. Thus there is an independent set I' in H' with $|I'| = |V(G)| - 1$. For v, we define that

$$L^*(v) = L(v) \setminus \{(v,k) : (v,k)(u,k) \in E(H), u \in N_G(v) \text{ and } (u,k) \in I'\}.$$

Since $|L(v)| \geq 4$ and v is a 3^--vertex, we have $|L^*(v)| \geq 1$. So we can pick a vertex $(v,c) \in L^*(v)$ such that $I' \cup \{(v,c)\}$ is an independent set of H with $|V(G)|$ vertices, a contradiction. □

Lemma 3.3. *G contains no separating 3-cycle.*

Proof. Let C be a separating 3-cycle in G. By the minimality of (G, C_0), any precoloring of C_0 can be extended to $G - int(C)$. After that, C is precolored, then again the coloring of C can be extended to $int(C)$. Thus, we get a DP-4-coloring of G, a contradiction. $\square$

Lemma 3.4. *(a) If a 3-face f is adjacent to a 4-face g in G, then f cannot be adjacent to a 3-face and g cannot be adjacent to a 3-or 4-face other than f.*

(b) If v is a 5^+-vertex with three consecutive incident 3-faces, or v is a 4-vertex with four incident 3-faces, then each of the 3-faces cannot be adjacent to other 3-faces.

(c) A 3-face f is not adjacent to a 5-face g.

Proof. (a) Let $f = uvw$ and $g = uwxy$. First we show that f cannot be adjacent to a 3-face. Suppose otherwise that f is adjacent to a 3-face $h = vzw$ by symmetry. Let $S = \{u, v, w, x, y\}$. If $z \notin S$, then $zwxyuv$ is a 6-cycle adjacent to a 3-cycle zwv, a contradiction. If $z \in S$, then $z = x$ or $z = y$. In the former case, $d(w) = 3$, contradicts Lemma 3.2. In the later case, $vy \in E(G)$. Since $d(u) \geq 4$, vuy is a separating 3-cycle, contradicts Lemma 3.3.

Next we show that g cannot be adjacent to another 3-face. Suppose otherwise that g is adjacent to a 3-face $h \neq f$. Since $\delta(G) \geq 4$, by symmetry h shares exactly one edge xw or xy with g. First we let $h = xwz$. If $z \notin S$, then $zxyuvw$ is a 6-cycle adjacent to a 3-cycle xwz, a contradiction. If $z \in S$, then $z = v$, which implies $d(w) = 3$, contradicts Lemma 3.2. Now let $h = zyx$. If $z \notin S$, then $zyuvwx$ is a 6-cycle adjacent to a 3-cycle zyx, a contradiction. If $z \in S$, then $z = v$, which implies vuy is a separating 3-cycle, contradicts Lemma 3.3.

Now we show that g cannot be adjacent to a 4-face. Suppose otherwise that g is adjacent to a 4-face h. Since $\delta(G) \geq 4$, by symmetry h shares exactly one edge xw or xy with g. First we let $h = xwzt$. Since $\delta(G) \geq 4$, $\{z, t\} \cap S = \emptyset$, but then $ztxyuw$ is a 6-cycle adjacent to a 3-cycle uvw, a contradiction. Now we let $h = yxzt$. Since $\delta(G) \geq 4$, z and t cannot be both in S. If $\{z, t\} \cap S = \emptyset$, then $ztyuwx$ is a 6-cycle adjacent to a 3-cycle uvw, a contradiction. If $|\{z, t\} \cap S| = 1$, then $t = v$ since $\delta(G) \geq 4$, which implies vuy is a separating 3-cycle, contradicts Lemma 3.3.

(b) First let v be a 5^+-vertex with three consecutive 3-faces $f_1 = uvw$, $f_2 = wvx$ and $f_3 = xvy$. Let $S = \{u, v, w, x, y\}$. Suppose otherwise that one of the three 3-faces is adjacent to another 3-face f_4. By symmetry we may assume that $f_4 = uzw$ or $f_4 = uvz$ or $f_4 = wzx$. Since $d(v) \geq 5$ and $\delta(G) \geq 4$, f_4 shares exactly one edge with the 3-faces. If $z \notin S$, then clearly there exists a 6-face adjacent to a 3-face, a contradiction. So we may assume that $z \in S$. Then $f_4 = uzw$ and $z = y$. But then wxy is a separating 3-cycle, contradicts Lemma 3.3. Now let v be a 4-vertex with $N(v) = \{v_i : 1 \leq i \leq 4\}$ and v is incident to four 3-faces $f_i = v_i v v_{i+1}$ (index module 4). By symmetry suppose that f_1 is adjacent to another 3-face $v_1 v_2 u$. If $u \notin N(v)$, then $uv_1v_4vv_3v_2$ is a 6-cycle adjacent to a 3-cycle uv_1v_2, a contradiction. If $u \in N(v)$, then by symmetry $u = v_3$. But then $v_1 v u$ is a separating 3-cycle, contradicts Lemma 3.3.

(c) Suppose otherwise that $f = xyz$ and $g = uvwxy$. If $z \notin S$, then $uvwxyz$ is a 6-cycle adjacent to a 3-cycle xyz, a contradiction. If $z \in S$, then we assume $z = u$ or $z = v$ by symmetry. In the former case, $d(y) = 2$, contradicts Lemma 3.2. In the later case, uvy is a separating 3-cycle, contradicts Lemma 3.3. $\square$

Corollary 3.5. *For $k \geq 5$, a k-vertex is incident to at most $k-2$ triangles.*

Lemma 3.6. *Two $(4,4,4)$-faces in $int(C_0)$ cannot share exactly one common edge in G.*

Proof. Suppose for a contradiction that $T_1 = uvx$ and $T_2 = uvy$ share a common edge uv. Let $S = \{u, v, x, y\}$ and $G' = G - S$. For each $v \in V(G')$, let $L'(v) = L(v)$ and let $H' = H \setminus \{L(w) : w \in S\}$. By the minimality of G, the graph G' has an (L', H')-coloring. Thus there is an independent set I' in H with $|I'| = |V(G)| - 4$.

For each $w \in S$, we define that

$$L^*(w) = L(w) \setminus \{(w,k) : (w,k)(u,k) \in E(H), u \in N(w) \text{ and } (u,k) \in I'\}.$$

Since $|L(v)| \geq 4$ for all $v \in V(G)$, we have

$$|L^*(u)| \geq 3,\ |L^*(v)| \geq 3,\ |L^*(x)| \geq 2,\ |L^*(y)| \geq 2.$$

So we can select a vertex (v,c) in $L^*(v)$ for v such that $L^*(x) \setminus \{(v,c) : (v,c)(x,c) \in E(H)\}$ has at least two available colors. Color y, u, x in order, we can find an independent set I^* with $|I^*| = 4$. So $I' \cup I^*$ is an independent set of H with $|I' \cup I^*| = |V(G)|$, a contradiction. $\square$

We are now ready to present a discharging procedure that will complete the proof of Theorem 1.2. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = 2d(v) - 6$, each face $f \neq D$ has an initial charge of $\mu(f) = d(f) - 6$ and $\mu(D) = d(D) + 6 = 9$. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = 0$.

Let $\mu^*(x)$ be the charge of $x \in V \cup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(D) > 0$.

The discharging rules:

(R1) Let f be a 3-face.

(R1.1) For a 4-vertex v where $v \notin V(C_0)$

$$w(v \to f) = \begin{cases} 1, & \text{if } v \text{ is a good, a bad, or a worse vertex of } f, \\ \frac{2}{3}, & \text{if } v \text{ is a worst vertex of } f, \\ 0.5, & \text{if } v \text{ is a hub vertex in induced } W_5. \end{cases}$$

(R1.2) For a 5^+-vertex v where $v \notin V(C_0)$

$$w(v \to f) = \begin{cases} 1.2, & \text{if } v \text{ is a good or a bad vertex of} f, \\ \frac{4}{3}, & \text{if } v \text{ is a worst vertex of } f, \\ 1.4, & \text{if } v \text{ is a worse vertex of } f. \end{cases}$$

(R1.3) Let g be a k-face adjacent to f with $k \geq 7$. Let E_0 be the subset of $E(g)$ such that each edge in E_0 has exactly one endpoint incident to f.

Let $w(g \to f) = (r + s + \frac{t}{2}) \times \frac{k-6}{k}$, where r is the number of common edges of g and f, s is the number of internal edges of g in E_0, and t is the number of non-internal edges of g in E_0.

(R2) Let f be a 4-face.

(R2.1) For a 4-vertex v where $v \notin V(C_0)$

$$w(v \to f) = \begin{cases} 0.4, & \text{if } v \text{ is a } (3,4,4,5)\text{-vertex}, \\ 0.6, & \text{if } v \text{ is incident to at least two } 5^+\text{-faces}, \\ 0.5, & \text{otherwise.} \end{cases}$$

(R2.2) For a 5^+-vertex v where $v \notin V(C_0)$ $w(v \to f) = 0.8$.

(R3) Let f be a 5-face.

$w(v \to f) = 0.2$ for each incident 4^+-vertex v.

(R4) A 5^+-vertex v where $v \notin V(C_0)$, distributes its remaining positive charge to all of its incident 3-faces within W_5 formed by four 3-faces equally. Then, redistribute the total of charges of 3-faces in the same cluster of adjacent 3-faces ($C(3,3,3)$ or W_5) equally among its 3-faces.

(R5) The outerface D gets $\mu(v)$ from each incident vertex and gives 2 to each 4-or 5-face or 3-face sharing exactly one vertex with D, $\frac{5}{2}$ to each 3-face sharing one edge with D.

It suffices to check that each $x \in V(G) \cup F(G)$ has nonnegative final charge and D has positive final charge. By (R5), we have $\mu^*(v) = 0$ for each $v \in V(C_0)$. Thus we only consider a vertex v where $v \notin V(C_0)$.

CASE 1: A 4-vertex v.

We use (R1.1), (R2.1), and (R3) to prove this case.

Assume v is incident to k 3-faces where $k \geq 2$. It follows from Lemmas 3.4(a) and (c) that the remaining incident faces of v are 7^+-faces. If v is incident to exactly two, three, or four 3-faces, respectively, then $\mu^*(v) = 2 - 2 \times 1 = 0$, $\mu^*(v) = 2 - 3 \times \frac{2}{3} = 0$, or $\mu^*(v) = 2 - 4 \times 0.5 = 0$, respectively.

Assume v is incident to exactly one 3-face. It follows from Lemmas 3.4(a) that v is not a $(3,4,4,4)$-vertex. This yields that v is a $(3,4,4,5)$-vertex, a $(3,4,4,6^+)$-vertex, a $(3,4^+,5^+,5^+)$-vertex, or v is incident to at most three faces (this situation happens only if some incident face is not a cycle), respectively. Thus $\mu^*(v) = 2 - 1 - 2 \times 0.4 - 0.2 = 0$, $\mu^*(v) = 2 - 1 - 2 \times 0.5 = 0$, $\mu^*(v) \geq 2 - 1 - 0.6 - 2 \times 0.2 = 0$, or $\mu^*(v) \geq 2 - 1 - 0.6 > 0$, respectively.

Assume v is not incident to any 3-face. Then v is a $(4,4,4,4^+)$-vertex, a $(4^+,4^+,5^+,5^+)$-vertex, or v is incident to at most three faces, respectively. Thus $\mu^*(v) \geq 2 - 4 \times 0.5 = 0$, $\mu^*(v) \geq 2 - 2 \times 0.6 - 2 \times 0.2 > 0$, or $\mu^*(v) \geq 2 - 3 \times 0.6 > 0$, respectively.

CASE 2: A 5-vertex v.

We use (R1.2), (R2.2), and (R3) to prove this case.

It follows from Corollary 3.5 that v is incident to at most three 3-faces.

Assume v is incident to exactly three 3-faces. It follows from Lemma 3.4 that remaining incident faces of v are 7^+-faces. If v is a worst vertex of some face, then $\mu^*(v) = 4 - 3 \times \frac{4}{3} = 0$, otherwise $\mu^*(v) = 4 - 2 \times 1.4 - 1.2 = 0$.

Assume v is incident to exactly two 3-faces. If v is incident to at most four faces, then $\mu^*(v) \geq 4 - 2 \times 1.4 - 2 \times 0.6 = 0$. If v is a worse vertex of some face and is incident to five faces, then v is a $(3,3,4^+,7^+,7^+)$-vertex by Lemma 3.4. Thus $\mu^*(v) \geq 4 - 2 \times 1.4 - 0.8 > 0$. If v is not a worse vertex of any face but is incident to five faces, then v is incident to at most one 4-face. Thus $\mu^*(v) \geq 4 - 2 \times 1.2 - 0.8 - 2 \times 0.2 > 0$.

Assume v is incident to exactly one 3-face. It follows from Lemma 3.4(a) that v is incident to at most three 4-faces. Thus $\mu^*(v) \geq 4 - 1.2 - 3 \times 0.8 - 0.2 > 0$.

Assume v is not incident to any 3-face. Then $\mu^*(v) \geq 4 - 5 \times 0.8 = 0$.

CASE 3: A 6-vertex v.

We use (R1.2), (R2.2), and (R3) to prove this case.

It follows from Corollary 3.5 that v is incident to at most four 3-faces. If v is incident to exactly four 3-faces, then the remaining incident faces are 7^+-faces by Lemma 3.4. Thus $\mu^*(v) \geq 6 - 4 \times 1.4 > 0$. If v is incident to exactly three 3-faces, then v is incident to at most one 4-face by Lemma 3.4(a). Thus $\mu^*(v) \geq 6 - 3 \times 1.4 - 0.8 - 2 \times 0.2 > 0$. If v is incident to at most two 3-faces, then $\mu^*(v) \geq 6 - 2 \times 1.4 - 4 \times 0.8 = 0$.

CASE 4: A 7^+-vertex v.

We use (R1.2), (R2.2), and (R3) to prove this case.

Assume v is a 7-vertex. It follows from Corollary 3.5 that v is incident to at most five 3-faces. If v is incident to exactly five 3-faces, then the remaining incident faces are 7^+-faces by Lemma 3.4. Thus $\mu^*(v) \geq 8 - 5 \times 1.4 > 0$. If v is incident to at most four 3-faces, then $\mu^*(v) \geq 8 - 4 \times 1.4 - 3 \times 0.8 = 0$.

Assume v is a k-vertex where $k \geq 8$. It follows from Corollary 3.5 that v is incident to at most $k-2$ 3-faces. Thus $\mu^*(v) \geq (2k-6) - (k-2) \times 1.4 - 2 \times 0.8 \geq 0$.

Let f be a face in G. Let $V(f) \cap V(D) \neq \emptyset$. If $d(f) = 3$, then f gets $\frac{5}{2}$ from D when f shares an edge with D, 2 from D when f shares exactly one vertex with D. Note that each vertex of f in $int(C_0)$ sends at least $\frac{1}{2}$ to f. So $\mu^*(f) \geq -3 + \min\{\frac{5}{2} + \frac{1}{2}, 2 + \frac{1}{2} \times 2\} = 0$. If $d(f) \in \{4, 5\}$, then it gets 2 from D. So $\mu^*(f) \geq d(f) - 6 + 2 \geq 0$. If $d(f) = 6$, then $\mu^*(f) = \mu(f) = 0$. If $d(f) \geq 7$, then $\mu^*(f) \geq (k-6) - k \times \frac{k-6}{k} = 0$. Now we may assume that $V(f) \cap V(D) = \emptyset$ in the rest of the paper.

CASE 5: A 3-face f.

We use (R1.1), (R1.2), (R1.3), (R4) to prove this case.

If f is not in $C(3,3,3)$, then $\mu^*(f) \geq -3 + 3 \times 1 = 0$.

Consider f is in $C(3,3,3)$ formed by three 3-faces, or induced W_5 formed by four 3-faces. Define $\mu(C(3,3,3)) := \mu(f_1) + \mu(f_2) + \mu(f_3) = -9$ where f_1, f_2, and f_3 are 3-faces in $C(3,3,3)$ and define $\mu^*(C(3,3,3)) := \mu^*(f_1) + \mu^*(f_2) + \mu^*(f_3)$. Similarly, we define $\mu(W_5) := -12$ and $\mu^*(W_5) := \mu^*(f_1) + \mu^*(f_2) + \mu^*(f_3) + \mu^*(f_4)$ where f_1, f_2, f_3, and f_4 are 3-faces in W_5.

Assume f is in $C(3,3,3)$ formed by three 3-faces, but not in W_5 formed by four 3-faces. Let $V(C(3,3,3)) = \{u, v, w, x, y\}$. Using (R4), it suffices to show that $\mu^*(C(3,3,3)) \geq 0$. It follows from Lemma 3.4 that each face adjacent to an internal face of $C(3,3,3)$ is a 7^+-face. If a worst vertex is a 5^+-vertex, then $\mu^*(C(3,3,3)) \geq -9 + 3 \times \frac{4}{3} + 6 \times 1 > 0$. If a worse vertex is a 5^+-vertex, then $\mu^*(C(3,3,3)) \geq -9 + 2 \times 1.4 + 4 \times 1 + 3 \times \frac{2}{3} + 5 \times \frac{1}{7} > 0$. If all worse and worst vertices are 4-vertices, then two bad vertices are 5^+-vertices, or one bad vertex is adjacent to both worse vertices by Lemma 3.6. In the former case, $\mu^*(C(3,3,3)) = -9 + 2 \times 1.2 + 4 \times 1 + 3 \times \frac{2}{3} + 5 \times \frac{1}{7} > 0$. In the later case, uvy is a separating 3-cycle, contradicts Lemma 3.3.

Assume f is in W_5 formed by four 3-faces. Let x be a hub and $uvwyu$ form an external cycle of W_5.

If u is adjacent to w, then uvw is a separating 3-cycle, contradicts Lemma 3.3. Thus u is not adjacent to w. Similarly, v is not adjacent to y. Together with Lemma 3.4, we have that each 3-face in W_5 is adjacent to a 7^+-face. Using Lemma 3.6 to subgraphs $C(3,3)$ of W_5, we obtain that at least two vertices of W_5 are 5^+-vertices. Thus $\mu^*(W_5) \geq -12+4\times 1.4+4\times 1+4\times 0.5+4\times \frac{1}{7} > 0$.

CASE 6: A 4-face f.

We use (R2.1) and (R2.2) to prove this case.

Let v_1, v_2, v_3, and v_4 be all vertices of f in a cyclic order. By Lemma 3.4(a), f is adjacent to at most one 3-face. If f is adjacent to a 3-face with an edge v_1v_2, then the remaining adjacent faces of f are 5^+-faces by Lemma 3.4(a). It follows that v_3 and v_4 are incident to at least two 5^+-faces. Thus $\mu^*(f) \geq -2 + 2 \times 0.6 + 2 \times 0.4 = 0$. If f is not adjacent to any 3-face, then each vertex of f is not a $(3,4,4,5)$-vertex. Thus $\mu^*(f) \geq -2 + 4 \times 0.5 = 0$.

CASE 8: A 5^+-face f.

If f is a 5-face, then $\mu^*(f) = -1 + 5 \times 0.2 = 0$ by (R3). If f is a 6-face, then $\mu^*(f) = \mu(f) = 0$. Consider the case that f is a k-face where $k \geq 7$. Note that each internal edge contributes at most $2 \times \frac{k-6}{k}$ to 3-faces in a discharging process, whereas each non internal edge contributes at most $\frac{k-6}{k}$ to 3-faces. This yields $\mu^*(f) \geq (k-6) - k \times \frac{k-6}{k} = 0$ by (R1.3).

CASE 9: The outerface D.

Let f_3', f' be the number of 3-faces sharing exactly one edge with D, 3-faces sharing exactly one vertex with D or 4-or 5-faces sharing vertices with D, respectively. Let $E(C_0, V(G) - C_0)$ be the set of edges between C_0 and $V(G) - C_0$ and let $e(C_0, V(G) - C_0)$ be its size. Then by (R5),

$$\mu^*(D) = 3 + 6 + \sum_{v \in C_0} (2d(v) - 6) - \frac{5}{2} f_3' - 2f' \tag{1}$$

$$= 9 + 2\sum_{v \in C_0} (d(v) - 2) - 2 \times 3 - \frac{5}{2} f_3' - 2f' \tag{2}$$

$$= 3 - \frac{1}{2} f_3' + 2(e(C_0, V(G) - C_0) - f_3' - f') \tag{3}$$

So we may think that each edge $e \in E(C_0, V(G) - C_0)$ carries a charge of 2. Since each 5^--face contains two edge in $E(C_0, V(G) - C_0)$, this implies that $e(C_0, V(G) - C_0) - f_3' - f' \geq 0$. Note that $f_3' \leq 3$. Then $\mu^*(D) > 0$ for any D.

This completes the proof.

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