

If statement (d) is true, then each row of  $U$  contains a pivot position and there can be no pivot in the augmented column. So  $A\mathbf{x} = \mathbf{b}$  has a solution for any  $\mathbf{b}$ , and (a) is true. If (d) is false, the last row of  $U$  is all zeros. Let  $\mathbf{d}$  be any vector with a 1 in its last entry. Then  $[U \ \mathbf{d}]$  represents an *inconsistent* system. Since row operations are reversible,  $[U \ \mathbf{d}]$  can be transformed into the form  $[A \ \mathbf{b}]$ . The new system  $A\mathbf{x} = \mathbf{b}$  is also inconsistent, and (a) is false. ■

**PRACTICE PROBLEMS**

- Let  $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$ ,  $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$ , and  $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$ . It can be shown that  $\mathbf{p}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ . Use this fact to exhibit  $\mathbf{b}$  as a specific linear combination of the columns of  $A$ .
- Let  $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ . Verify Theorem 5(a) in this case by computing  $A(\mathbf{u} + \mathbf{v})$  and  $A\mathbf{u} + A\mathbf{v}$ .

**1.4 EXERCISES**

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing  $A\mathbf{x}$ . If a product is undefined, explain why.

- $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of  $A\mathbf{x}$  to write the matrix equation as a vector equation, or vice versa.

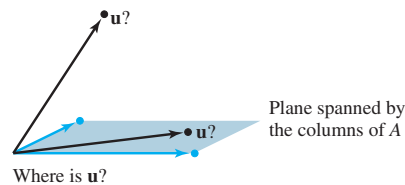
- $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$
- $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$
- $z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

- $5x_1 + x_2 - 3x_3 = 8$   
 $2x_2 + 4x_3 = 0$
- $4x_1 - x_2 = 8$   
 $5x_1 + 3x_2 = 2$   
 $3x_1 - x_2 = 1$

Given  $A$  and  $\mathbf{b}$  in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation  $A\mathbf{x} = \mathbf{b}$ . Then solve the system and write the solution as a vector.

- $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$
- Let  $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$  and  $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$ . Is  $\mathbf{u}$  in the plane in  $\mathbb{R}^3$  spanned by the columns of  $A$ ? (See the figure.) Why or why not?



- Let  $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$  and  $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$ . Is  $\mathbf{u}$  in the subset of  $\mathbb{R}^3$  spanned by the columns of  $A$ ? Why or why not?

15. Let  $A = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ . Show that the equation  $A\mathbf{x} = \mathbf{b}$  does not have a solution for all possible  $\mathbf{b}$ , and describe the set of all  $\mathbf{b}$  for which  $A\mathbf{x} = \mathbf{b}$  does have a solution.

16. Repeat the requests from Exercise 15 with

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Exercises 17–20 refer to the matrices  $A$  and  $B$  below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

17. How many rows of  $A$  contain a pivot position? Does the equation  $A\mathbf{x} = \mathbf{b}$  have a solution for each  $\mathbf{b}$  in  $\mathbb{R}^4$ ?
18. Can every vector in  $\mathbb{R}^4$  be written as a linear combination of the columns of the matrix  $B$  above? Do the columns of  $B$  span  $\mathbb{R}^4$ ?
19. Can each vector in  $\mathbb{R}^4$  be written as a linear combination of the columns of the matrix  $A$  above? Do the columns of  $A$  span  $\mathbb{R}^4$ ?
20. Do the columns of  $B$  span  $\mathbb{R}^4$ ? Does the equation  $B\mathbf{x} = \mathbf{y}$  have a solution for each  $\mathbf{y}$  in  $\mathbb{R}^4$ ?

21. Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ . Does

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  span  $\mathbb{R}^4$ ? Why or why not?

22. Let  $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$ . Does

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  span  $\mathbb{R}^3$ ? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. The equation  $A\mathbf{x} = \mathbf{b}$  is referred to as a *vector equation*.  
 b. A vector  $\mathbf{b}$  is a linear combination of the columns of a matrix  $A$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution.  
 c. The equation  $A\mathbf{x} = \mathbf{b}$  is consistent if the augmented matrix  $[A \ \mathbf{b}]$  has a pivot position in every row.  
 d. The first entry in the product  $A\mathbf{x}$  is a sum of products.  
 e. If the columns of an  $m \times n$  matrix  $A$  span  $\mathbb{R}^m$ , then the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for each  $\mathbf{b}$  in  $\mathbb{R}^m$ .  
 f. If  $A$  is an  $m \times n$  matrix and if the equation  $A\mathbf{x} = \mathbf{b}$  is inconsistent for some  $\mathbf{b}$  in  $\mathbb{R}^m$ , then  $A$  cannot have a pivot position in every row.

24. a. Every matrix equation  $A\mathbf{x} = \mathbf{b}$  corresponds to a vector equation with the same solution set.  
 b. If the equation  $A\mathbf{x} = \mathbf{b}$  is consistent, then  $\mathbf{b}$  is in the set spanned by the columns of  $A$ .  
 c. Any linear combination of vectors can always be written in the form  $A\mathbf{x}$  for a suitable matrix  $A$  and vector  $\mathbf{x}$ .  
 d. If the coefficient matrix  $A$  has a pivot position in every row, then the equation  $A\mathbf{x} = \mathbf{b}$  is inconsistent.  
 e. The solution set of a linear system whose augmented matrix is  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$  is the same as the solution set of  $A\mathbf{x} = \mathbf{b}$ , if  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ .  
 f. If  $A$  is an  $m \times n$  matrix whose columns do not span  $\mathbb{R}^m$ , then the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $\mathbf{b}$  in  $\mathbb{R}^m$ .

25. Note that  $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$ . Use this

fact (and no row operations) to find scalars  $c_1, c_2, c_3$  such that  $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$ .

26. Let  $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ , and  $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ . It can be shown that  $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$ . Use this fact (and no row operations) to find  $x_1$  and  $x_2$  that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

27. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols  $\mathbf{v}_1, \mathbf{v}_2, \dots$  for the vectors and  $c_1, c_2, \dots$  for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

28. Let  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ , and  $\mathbf{v}$  represent vectors in  $\mathbb{R}^5$ , and let  $x_1, x_2$ , and  $x_3$  denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$$

29. Construct a  $3 \times 3$  matrix, not in echelon form, whose columns span  $\mathbb{R}^3$ . Show that the matrix you construct has the desired property.
30. Construct a  $3 \times 3$  matrix, not in echelon form, whose columns do *not* span  $\mathbb{R}^3$ . Show that the matrix you construct has the desired property.
31. Let  $A$  be a  $3 \times 2$  matrix. Explain why the equation  $A\mathbf{x} = \mathbf{b}$  cannot be consistent for all  $\mathbf{b}$  in  $\mathbb{R}^3$ . Generalize your argument to the case of an arbitrary  $A$  with more rows than columns.

42 CHAPTER 1 Linear Equations in Linear Algebra

32. Could a set of three vectors in  $\mathbb{R}^4$  span all of  $\mathbb{R}^4$ ? Explain. What about  $n$  vectors in  $\mathbb{R}^m$  when  $n$  is less than  $m$ ?
33. Suppose  $A$  is a  $4 \times 3$  matrix and  $\mathbf{b}$  is a vector in  $\mathbb{R}^4$  with the property that  $A\mathbf{x} = \mathbf{b}$  has a unique solution. What can you say about the reduced echelon form of  $A$ ? Justify your answer.
34. Let  $A$  be a  $3 \times 4$  matrix, let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be vectors in  $\mathbb{R}^3$ , and let  $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ . Suppose  $\mathbf{v}_1 = A\mathbf{u}_1$  and  $\mathbf{v}_2 = A\mathbf{u}_2$  for some vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  in  $\mathbb{R}^4$ . What fact allows you to conclude that the system  $A\mathbf{x} = \mathbf{w}$  is consistent? (Note:  $\mathbf{u}_1$  and  $\mathbf{u}_2$  denote vectors, not scalar entries in vectors.)
35. Let  $A$  be a  $5 \times 3$  matrix, let  $\mathbf{y}$  be a vector in  $\mathbb{R}^3$ , and let  $\mathbf{z}$  be a vector in  $\mathbb{R}^5$ . Suppose  $A\mathbf{y} = \mathbf{z}$ . What fact allows you to conclude that the system  $A\mathbf{x} = 5\mathbf{z}$  is consistent?
36. Suppose  $A$  is a  $4 \times 4$  matrix and  $\mathbf{b}$  is a vector in  $\mathbb{R}^4$  with the property that  $A\mathbf{x} = \mathbf{b}$  has a unique solution. Explain why the columns of  $A$  must span  $\mathbb{R}^4$ .

[M] In Exercises 37–40, determine if the columns of the matrix span  $\mathbb{R}^4$ .

37. 
$$\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$$

38. 
$$\begin{bmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{bmatrix}$$

39. 
$$\begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ -8 & 4 & -6 & -10 & -3 \\ -7 & 11 & -5 & -1 & -8 \\ 3 & -1 & 10 & 12 & 12 \end{bmatrix}$$

40. 
$$\begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ -7 & -3 & -4 & 6 & -9 \\ 11 & 5 & 6 & -9 & -3 \\ -3 & 4 & -7 & 2 & 7 \end{bmatrix}$$

41. [M] Find a column of the matrix in Exercise 39 that can be deleted and yet have the remaining matrix columns still span  $\mathbb{R}^4$ .
42. [M] Find a column of the matrix in Exercise 40 that can be deleted and yet have the remaining matrix columns still span  $\mathbb{R}^4$ . Can you delete more than one column?

**SG** Mastering Linear Algebra Concepts: Span 1–18

**WEB**

SOLUTIONS TO PRACTICE PROBLEMS

1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3 \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

which expresses  $\mathbf{b}$  as a linear combination of the columns of  $A$ .

2. 
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$A(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 + 20 \\ 3 + 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$$

$$A\mathbf{u} + A\mathbf{v} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 11 \end{bmatrix} + \begin{bmatrix} 19 \\ -4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$$