Every planar graph without 5-cycles nor adjacent triangles nor adjacent 4-cycles is $(2, 0, 0)$-colorable

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Abstract

In 1976, Steinberg conjectured that every planar graph without 4-cycles and 5-cycles is 3-colorable. Borodin and Raspaud (2003) further conjectured that every planar graph without 5-cycles and K_4^- is 3-colorable. In 2016, these two conjectures are disproved by Cohen–Addad and others. Now in this paper, we prove a relaxation of Strong Bordeaux Conjecture that every planar graph without 5-cycles and adjacent triangles and adjacent 4-cycles is $(2, 0, 0)$-colorable which improves the results of Chen, Wang, Liu and Xu (2016) and of Liu, Li and Yu (2015).

1 Introduction

It is well-known that deciding whether a planar graph is properly 3-colorable is a NP-complete problem. Grötzsch [9] proved the famous theorem that every triangle-free planar graph is 3-colorable. Steinberg in 1976 made the following conjecture [16].

conj1 **Conjecture 1.1 (Steinberg, [16])** *All planar graphs without 4-cycles and 5-cycles are 3-colorable.*

This conjecture was disproved by Cohen–Addad *et al.* [7] recently. However, Erdős suggested to find a constant c such that a planar graph without cycles of length from 4 to c is 3-colorable. The best constant so far is $c = 7$, found by Borodin, Glebov, Raspaud, and Salavatipour [4].

A graph is $(c_1, c_2, \ldots, c_k)$-*colorable* if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that for every i, the subgraph $G[V_i]$ has maximum degree at most c_i, where $1 \le i \le k$. Improper colorability of graphs has been extensively studied. For more results, see [12, 6, 19, 20] and the survey by Borodin [1].

As usual, a 3-cycle is also called a *triangle*. Havel in [10] asked if each planar graph with large enough distances between triangles is $(0, 0, 0)$-colorable. This was resolved by Dvořák, Král and Thomas [8]. We say that two cycles are *adjacent* if they have at least one edge in common and *intersecting* if they have at least one common vertex. A graph contains a pair of adjacent triangles if and only if it contains a K_4^- as a subgraph. Borodin and Raspaud in 2003 made the following two conjectures, which have common features with Havel's and Steinberg's 3-color problems.

con2 **Conjecture 1.2 (Bordeaux Conjecture, [5])** *Every planar graph without* 5*-cycles and intersecting* 3*-cycles and is* 3*-colorable.*

con3 **Conjecture 1.3 (Strong Bordeaux Conjecture, [5])** *Every planar graph without* 5*-cycles and* K_4^- *is* 3*-colorable.*

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Let G be a plane graph. Denote by d^{∇} the minimum distance between two triangles in G. A relaxation of the Bordeaux Conjecture with $d^{\nabla} \geq 4$ was confirmed by Borodin and Rauspaud [5], and the result was improved to $d^{\nabla} \geq 3$ by Borodin and Glebov [2] and, independently, by Xu [17]. Borodin and Glebov [3] further improved the result to $d^{\nabla} \geq 2$.

For relaxations of Conjecture 1.2, Liu, Li and Yu [14, 15] proved that every planar graph without 5-cycles and intersecting 3-cycles is $(2,0,0)$-colorable and $(1,1,0)$-colorable. Conjecture 1.3 was also disproved by Cohen–Addad *et al.* [7] recently. Xu [18] showed that every graph without 5-cycles nor K_4^- is $(1,1,1)$-colorable, which was improved to be $(1,1,0)$-colorable by Huang, Li and Yu [11]. One may naturally ask the following question.

prob1 **Problem 1.4** *Every graph without 5-cycles nor K_4^- is $(1,0,0)$-colorable.*

On the other hand, Chen *et al.* [6] proved that every planar graph without 5-cycles nor 4-cycle is $(2,0,0)$-colorable. Motivated by Problem 1.4 and the result of Chen *et al.* [6], we consider $\mathscr{G}$, the family of plane graph without 5-cycle nor two adjacent 3-cycles nor two adjacent 4-cycles. Here is our main result.

th2 **Theorem 1.5** *Every planar graph without 5-cycles, or K_4^-, or adjacent 4-cycles is (2,0,0)-colorable.*

We actually prove something stronger. Let G be a plane graph and H be an induced subgraph of G. We call (G,H) is *superextendable* if every $(2,0,0)$-coloring of H can be extended to a $(2,0,0)$-coloring of G such that the vertices in $G-H$ have different colors from their neighbors in H. Let $G \in \mathscr{G}$. An induced k-cycle C of G, where $k \in \{3,7,9\}$, is *bad* if (G,C) is not superextended. Thus, the outer cycle in B_i with $i \in [6]$ shown in Figure 1 is a bad cycle. An induced k-cycle is *good* if it is not bad, where $k \in \{3,7,9\}$.

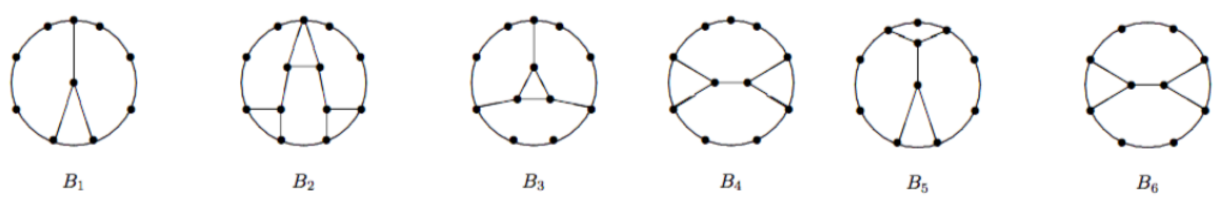

Figure 1: Six bad cycles fig1

th1 **Theorem 1.6** *Every triangle or induced 7-cycle or induced good 9-cycle of planar graph in $\mathscr{G}$ is superextendable.*

Proof of Theorem 1.5 via Theorem 1.6: Let G be a graph in $\mathscr{G}$. If G is triangle-free, then G is 3-colorable by the Gröztch Theorem, and is naturally $(2,0,0)$-colorable. Thus, assume that G has a triangle. Then every $(2,0,0)$-coloring of this triangle can be superextended to the whole graph G by Theorem 1.6. So, Theorem 1.5 follows.□

2 Reducible Configurations

All the graphs considered in this paper are finite and simple. For each $v \in V(G)$, we use $d(v)$ to denote the degree of v, and $N(v)$ to denote the vertex set of neighbors of v. For a face f of G, we use $V(f)$ to denote the vertex set on f and $d(f)$ to denote the degree of f. A *k-vertex* (*k^+-vertex*, *k^--vertex*) is a vertex of degree k (at least k, at most k). The same notation will apply to faces and cycles. For a face f of G, we write $f=[u_1u_2\ldots u_k]$ if $u_1,u_2,\ldots,u_k$ are consecutive vertices on f in a cyclic order, and we say that f is a $(d(u_1),d(u_2),\ldots,d(u_k))$*-face*. A face f is a *pendent 3-face* of vertex v if v is not on f but is adjacent to some 3-vertex on f. A *pendent neighbor*, denoted by v', of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face. If an edge uv is not an edge of any triangle, then u is called an *isolated neighbor* of v. A vertex is *k-triangular* if it is incident with k triangles. Note that G has no adjacent triangles. If a vertex is k-triangular, then it has degree at least $2k$. The boundary of the unbounded face of a plane graph is called the *outer cycle* if it is a cycle.

Let C be a cycle of a plane graph G. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C, respectively. The cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$, and is called a *nonseparating cycle* otherwise. We still use C to denote the set of vertices of C.

Let $S_1, S_2, \ldots, S_l$ be pairwise disjoint subsets of $V(G)$. We use $G[S_1, S_2, \ldots, S_l]$ to denote the graph obtained from G by identifying all the vertices in S_i to a single vertex for each $i \in \{1, 2, \ldots, l\}$. Let v_{xy} be the new vertex by identifying x and y in G.

A vertex v is *properly colored* if all neighbors of v have different colors from v. A vertex v is *nicely colored* if it shares a color (say i) with at most $\max\{s_i - 1, 0\}$ neighbors, where s_i is the deficiency allowed for color i.

Let (G, C_0) be a minimum counterexample to Theorem 1.6 with minimum $\sigma(G) = |V(G)| + |E(G)|$, where C_0 is an outer cycle of the unbounded face of G that is precolored and we further assume that C_0 is an induced outer cycle. Some earlier results from [14, 6] are stated in the following lemmas since the results of our lemmas can be proved similarly from their proofs.

le1 **Lemma 2.1** *Each of the following is true.*

(1) *Every vertex not on C_0 is a 3^+-vertex.*
(2) *A 3-face cannot share a common edge with a 4-face in G.*
(3) *No two 3-faces in G are adjacent.*
(4) *(Lemma 3.2 [14]) There is no separating good induced k-cycle, where $k \in \{3, 7, 9\}$.*
(5) *(Lemma 3.8 [14]) A 3-vertex must be adjacent to a 5^+-vertex or a vertex on C_0. Consequently, the pendent neighbor of the 3-vertex of a $(3,4,4)$-face in $int(C_0)$ is a 5^+-vertex or a vertex in C_0.*
(6) *(Lemma 3.9 [14]) The pendent neighbor of the 3-vertices in a $(3,3,5^-)$-face in $int(C_0)$ is a 5^+-vertex or a vertex on C_0.*

An edge $e = uv$ is called a (k_1, k_2)-*chord* of cycle C if $u, v \in C$ and the two paths between u, v on C and e form two cycles of lengths k_1 and k_2, respectively. Since G has no adjacent cycles of length at most five, the following remark is straightforward.

rmk1 **Remark 2.1** *Let C be a cycle in G.*

(1) If $|C| = 3, 4, 6$, then C has no chord.

(2) If $|C| = 7$, then C has at most one $(3,6)$-chord.

(3) If $|C| = 9$, then C has at most three chords. If C has one chord, then it has a $(4^-, k)$-chord, where $k \in \{7, 8\}$. If C has two chords, then C has either a $(4,7)$-chord and a $(3,8)$-chord or two $(3,8)$-chords. If C has three chords, then it has either a $(4,7)$-chord and two $(3,8)$-chords or three $(3,8)$-chords.

le00 **Lemma 2.2** *Let $C = u_1u_2 \ldots u_k$ be a cycle of G.*

- *If $k = 4, 6$, then $int(C) = \emptyset$. So there is no separating 4- or 6-cycle.*
- *Let $k = 8$. If $int(C) \neq \emptyset$, then C is the outer face in B_6 in Fig. 1. If $int(C) = \emptyset$, then C has at most two chords. Moreover, if C has one chord, then it is a $(3,7)$-chord or a $(4,6)$-chord; if C has two chords, then they are $(3,7)$-chords.*

Proof. Suppose otherwise that $k \in \{4, 6\}$ and $int(D) \neq \emptyset$. Let $G' = G - int(C)$. By the minimality of G, (G', C_0) can be superextended to a $(2,0,0)$-coloring of G'. It follows that C has a $(2,0,0)$-coloring. If $k = 4$, let $C' = u_1w_1w_2w_3u_2u_3u_4$ and properly color w_1, w_2, w_3. If $k = 6$, let $C' = u_1w_1u_2u_3u_4u_5u_6$ and properly color w_1. Then C' is a precolored 7-cycle. By minimality of G, $(C' \cup int(C), C')$ is superextendable and thus (G, C) is $(2,0,0)$-colorable, a contradiction.

Now let $k = 8$ and assume that C is not the outer face in B_6. Assume first that $int(C) \neq \emptyset$. Then $(G - \int(C), C_0)$ is $(2,0,0)$-colorable and we obtain a coloring of vertices of C. Let $C' = u_1w_1u_2u_3u_4u_5u_6u_7u_8$

and properly color w_1. By minimality of G, if C' is not one of outer faces in $B_1, \ldots, B_6$ in Figure 1, then $(C' \cup int(C), C')$ is superextendable and thus (G, C) is $(2,0,0)$-colorable, a contradiction. Thus, C' is one of the outer faces in $B_1, \ldots, B_5$. Since G has no 5-cycle, w_1 cannot be on a 6-cycle. Thus, C' can only be in B_4 or B_5, and before adding w_1, C must be the outer face in B_6, a contradiction. Now let $int(C) = \emptyset$. If C has a $(3,7)$-chord, then another possible chord of C can only be a $(3,7)$-chord; if C has no $(3,7)$-chord, then the only possible chord is a $(4,6)$-chord. ■

2p **Lemma 2.3** *If $P = xyz$ is a path with $x, z \in C_0$ and $y \in int(C_0)$, then $xz \in E(G)$.*

Proof. Suppose otherwise that $xz \notin E(G)$. Let P_1 and P_2 be the two paths between x and z on C_0, $C_i = P_i \cup P$ and $G_i = int(C_i)$ for $i = 1, 2$. Then $|C_i| \geq 4$ for $i = 1, 2$. We may assume that $4 \leq |C_1| \leq |C_2|$. Since $|C_1| + |C_2| \leq 13$ and G has no 5-cycles, $|C_1| \in \{4, 6\}$.

Assume first that $|C_1| = 4$. By Remark 2.1(1) and Lemma 2.2, C_1 is a 4-face. Since $|C_0| \in \{3, 7, 9\}$, $|C_2| \in \{7, 9\}$. By Lemma 2.1(1), $d(y) \geq 3$. Let y' be a neighbor of y rather than x and z. Since G contains no 5-cycle or K_4^-, $y' \notin C_0$. So all neighbors of y are in $int(C_2)$. By Lemma 2.1(4), C_2 must be a bad 9-cycle. If $d(y) \geq 4$, then C_2 is the outer face in B_2 in Figure 1, which implies that a 3-neighbor $y' \in int(C_0)$ of y has no 5^+-neighbors in $int(C_0)$, contrary to Lemma 2.1(5). Thus, $d(y) = 3$. Since G has no 5-cycle or K_4^-, C_2 is not in B_2, B_4, B_5 or B_6. If y is in B_1, then C_0 is the outer face of B_5, a contradiction. If y is in B_3, then G contains a 3-vertex in $int(C_0)$ that has no 5^+-neighbors, contrary to Lemma 2.1(5).

Thus, $|C_1| = 6$. Since $|C_0| \in \{7, 9\}$, $|C_2| = 7$. Since G contains no separating 7-cycle, y has no neighbor in $int(C_2)$. By Lemma 2.2, y has no neighbor in $int(C_1)$. Since $d(y) \geq 3$, a neighbor (say y') of y must be on P_1 or P_2. But since G has no 5-cycle or K_4^-, yy' must be a $(3,6)$-chord on C_2, which implies a B_4 containing C_0 as outer face, a contradiction. ■

3p **Lemma 2.4** *If $P = wxyz$ is a path with $w, z \in C_0$ and $x, y \in int(C_0)$, then $wz \in E(G)$.*

Proof. Suppose to the contrary that $wz \notin E(G)$. Let P_1 and P_2 be the two paths between w and z on C_0, $C_i = P_i \cup P$, and $G_i = int(C_i)$, where $i = 1, 2$. By Lemma 2.1(1), $d(x) \geq 3$ and $d(y) \geq 3$ and let x' be a neighbor of x other than w, y and y' be a neighbor of y other than x, z. Then $|C_i| \geq 6$ for $i = 1, 2$ since G has no 5-cycles. We may assume that $6 \leq |C_1| \leq |C_2|$. Since $|C_1| + |C_2| = |C_0| + 6 \leq 15$, $|C_1| \in \{6, 7\}$. We consider the following two cases.

We first assume that $|C_1| = 6$. By Lemma 2.2, C_1 is a face. In this case, $|C_2| \in \{6, 7, 9\}$. If $|C_2| = 6$, then C_2 is also a 6-faces by Lemma 2.2. It follows that $d(x) = d(y) = 2$, contrary to Lemma 2.1(1). If $|C_2| = 7$, then C_2 has at most one $(3,6)$-chord by Lemma 2.1(4). It follows that either $d(x) = 2$ or $d(y) = 2$, contrary to Lemma 2.1(1). Finally, assume that $|C_2| = 9$. If C_2 is good, then C_2 has two $(2,8)$-chords. In this case, C_0 has one bad partition of B_4 and B_5, a contradiction. Thus, assume that C_2 has a bad partition. Since both x and y are 3^+-vertices, C_2 has no bad partition B_3. If C_2 has the bad partition B_1, then C_0 has a bad partition B_3, a contradiction. If C_2 has one bad partition of B_2, B_4, B_5, then G has a 3-vertex x' in $int(C_0)$ which has no 5^+-neighbor in $int(C_0)$ nor a neighbor on C_0, contrary to Lemma 2.1(2).

We now assume that $|C_1| = 7$. Then C_1 is good and at most one $(3,6)$-chord. In this case, $|C_2| = 8$. By Lemma 2.2, C_2 has a bad partition B_6 or C_2 has at most two chords. In the former case, if C_1 has no chord, then G has a 3-vertex x' in $int(C_0)$ which has no 5^+-neighbor in $int(C_0)$ nor a neighbor on C_0, contrary to Lemma 2.1(2); if C_1 has one $(3,6)$-chord, then C_0 has the bad partition B_2, a contradiction. In the latter case, since both x and y are 3^+-vertices and C_1 has at most one $(3,6)$-chord and C_2 has at most two $(3,6)$, C_0 has the bad partition B_4, a contradiction. ■

4p **Lemma 2.5** *(1) If $P = vwxyz$ is a path with $v, z \in C_0$ and $w, x, y \in int(C_0)$, then P and one of the two paths of C_0 between v and z form a k-cycle, where $k \in \{6, 7\}$.*

(2) ([6])If $P = uvwxyz$ is a path with $u, z \in C_0$ and $v, w, x, y \in int(C_0)$, then P and one of the two paths between u and z form a k-cycle, where $k \in \{6, 7, 8, 9\}$.

Proof. (1) Let P_1 and P_2 be the two paths between w and z on C_0. For $i = 1, 2$, let $C_i = P_i \cup P$, and $G_i = int(C_i)$. By way of contradiction, we assume that $8 \leq |C_1| \leq |C_2|$. By Lemma 2.4, we may assume that $vx, wy, xz, vy, wz \notin E(G)$. Since $|C_1| + |C_2| \leq |C_0| + 8 \leq 17$, $|C_1| = 8$ and $|C_2| \in \{8, 9\}$. By Lemma 2.1(1), $\min\{d(x), d(y), d(w)\} \geq 3$, let x', y', w' be a neighbor of x, y, z not in $\{v, w, x, y, z\}$, respectively. By Lemma 2.2, C_1 is the outer face in B_6 or C_1 has at most two chords.

First let $|C_2| = 8$. Then C_1 or C_2 contains x', y' or w', so at least one of them is the outer face in B_6. If x' is not on C_0, then x' is in a B_6, but then x' has no 5^+-neighbors. So x' must be a vertex on C_0, and by Lemma 2.2, x' can only be on a $(4, 6)$-chord. We may assume that C_1 contains this $(4, 6)$-chord. This implies that w' cannot be on a $(3, 7)$- or $(4, 6)$-chord, so it is in $int(C_2)$, then C_2 is the outer face of a B_6. But then G contains a 5-cycle which is formed by the 4-cycle containing x' and a triangle of B_6, a contradiction.

Thus, assume that $|C_2| = 9$. First let x' be a vertex on C_0. Then x' is on $(4, 6)$-chord, we may assume that x' is next to v on C_0. If $x' \in P_2$, then ww' cannot be on a chord of C_1, so $w' \in int(C_1)$, thus C_1 must be the outer face of a B_6. It follows that vw is on a triangle, which together with the 4-cycle containing xx' forms a 5-cycle, a contradiction. If $x' \in P_1$, then $C' = x', x, y, z, P_2, v$ is a 9-cycle so that $w \in int(C')$. Then C' must be the outer face of B_i for some $i \in [5]$, which contains a 4-cycle $wxx'v$. So C' is on B_5. Now y' cannot be on C_0 or $int(C_0)$, a contradiction. So we may assume that $x' \in int(C_0)$. If $x' \in int(C_1)$, then C_1 is the outer face of B_6, so $x \in int(C_0)$ and x has no 5^+-neighbors or neighbors on C_0, a contradiction. So let $x' \in int(C_2)$. Then C_2 is the outer face of B_i for some $i \in [5]$. Then again, x has no 5^+-neighbors or neighbors on C_0, a contradiction. ■

We now give two useful technique lemmas on identifying vertices.

le12 **Lemma 2.6** *Let the neighbors of a k-vertex $v \in int(C_0)$ be $v_1, v_2, \ldots, v_k$ in the clockwise order in the embedding of G with $v_{k+1} = v_1$ and $k \geq 4$. Let v_i and v_j be two nonconsecutive neighbors of v. If G' is the graph obtained by identifying v_i and v_j of $G - v$, where $i < j$, then $G' \in \mathcal{G}$.*

Proof. Since v_i and v_j are two nonconsecutive neighbors of v, v_i is not adjacent to v_j since G has no separating 3-cycle by Lemma 2.1. By Lemma 2.3, at least one of v_i and v_j is not on C_0. Thus, we do not identify two vertices of C_0. We first show that G' has no chord. Suppose otherwise that G' has a chord. Then the chord contains the vertex $v_{v_i v_j}$. This implies that there is a 3-path $v_i v v_j u$ (or $v_j v v_i u$), where v_i (or v_j) and u are on C_0. By Lemma 2.4, $v_i v_j u$ (or $v_j v v_i u$) is a 4-cycle and v_{i+1} is it, contrary to Lemma 2.2.

Finally, we show that no k-cycle with $k \leq 5$ contains the vertex $v_{v_i v_j}$. Suppose otherwise. If $k = 3$, G contains a separating 5-cycle containing $v_i v v_j$, contrary to Lemma 2.1. If $k = 4$, then G contains a 6-cycle containing $v_i v v_j$ with v_{i+1} in it, contrary to Lemma 2.2. If $k = 5$, G contains a separating 7-cycle containing $v_i v v_j$, contrary to Lemma 2.1. ■

le12a **Lemma 2.7** *Let $v \in int(C_0)$ be a 3-triangular 7-vertex or a 4-triangular 8-vertex with $N[v_0] \subseteq int(C_0)$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of $v \in int(C_0)$ in the clockwise order in the embedding of G with $v_{k+1} = v_1$. Let v_i and v_j be two nonconsecutive 3-neighbors of v such that v_i and v_j are on two distinct 3-faces, let v_i' and v_j' be the outer neighbors of v_i and v_j, respectively. Let G' be the graph obtained by identifying v_i' and v_j' of $G - \{v, v_i, v_j\}$. Then $G' \in \mathcal{G}$ if one of the following holds.*

(1) $k = 7, 8$, $j = i + 4$ and $[v_i v v_{i+1}]$ and $[v_j v v_{j+1}]$ are both 3-faces.

(2) $k = 7$, $j = i + 3$ and $[v_i v v_{i+1}]$ and $[v_{j-1} v v_j]$ are both 3-faces.

Proof. We first show that at most one of v_i', v_j' is on C_0. For otherwise, by Lemma 2.5 (1), there is a 6- or 7-cycle containing v_i' and v_j', and the cycle is separating, a contradiction to Lemma 2.1(4) and 2.2. This implies that we do not destroy the cycle C_0 by identifying vertices.

Now we show that the identification creates none of the following: a chord on C_0, a 5-cycle, a K_4^-, or two adjacent 4-cycles. Suppose otherwise. We first claim that there is k-cycle of length at most 9 containing v_i', v_i, v, v_j, v_j'. Indeed, if a chord on C_0 is created, then we may assume that $v_i' \in C_0$ and v_j' is adjacent to a vertex, say u, on C_0. By Lemma 2.3, there is k-cycle of length in $\{6, 7, 8, 9\}$ containing $v_i', v_i, v, v_j, v_j', u$. If G' contains a 5-cycle, a K_4^-, or two adjacent 4-cycles. So the resulting vertex v' must be a vertex on a 5-cycles, K_4^-, or adjacent 4-cycle in G'. It follows that there is a path, say P of length at most five between v_i' and v_j' in $G - \{v, v_i, v_j\}$. Then P together with v, v_i, v_j, v_i', v_j' forms a cycle of length at most 9 in G.

Let the k-cycle with $k \geq 9$ be C. Then C is a separating cycle. By Lemma 2.1 (4) and Lemma 2.2, C must be a 8-cycle that is the outer face of B_6 or is a 9-cycle that is the outer face of B_i with $i \in [5]$. Since v is on C and v has degree at least four inside or outside C, C can only be the outer face of B_2, in which v is the 4-vertex that is adjacent to a triangle. But the conditions we have chosen forbid this possibility.

Therefore, $G' \in \mathcal{G}$. ■

From now on, let $F_k = \{f : f \text{ is a } k\text{-face and } b(f) \cap C_0 = \emptyset\}$, $F'_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 1\}$, and $F''_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 2\}$, $F'''_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 3\}$.

le1a **Lemma 2.8** *(1) If $f = [uxvy]$ is a 4-face and $|\{u, v\} \cap C_0| \leq 1$, then $G[\{u, v\}] \in \mathscr{G}$.*

(2) There is no 4-face from F'_4.

(3) There is no $(4^-, 3^+, 4^-, 3^+)$-face f with $b(f) \cap C_0 = \emptyset$.

Proof. (1) Since G has no 5-cycle, there is no 3-path joining u and v. It follows that no new triangle is created in $G[\{u, v\}]$ and hence $G[\{u, v\}]$ has no K_4^-. By Lemma 2.2, there is no 4-path joining u and v. Thus, no new 4-cycle is created in $G[\{u, v\}]$ and hence $G[\{u, v\}]$ has no adjacent 4-cycles. If $G[\{u, w\}]$ has a 5-cycle, then G has a 5-path P' joining u and v. If one of x and y is in P', then $b(f) \cup P'$ has a 5-cycle, a contradiction. So, $x, y \notin V(P')$, and hence either $P' \cup uxv$ or $P' \cup uyv$ is a separating 7-cycle; both contradict Lemma 2.1 (4). Therefore, $G[\{u, v\}] \in \mathscr{G}$.

(2) Suppose otherwise that $f = [uvwx]$ is a 4-face from F'_4 such that $b(f) \cap C_0 = \{u\}$. Let $C_0 = [v_1 v_2 \ldots v_k]$, where $k \in \{3, 7, 9\}$. We assume, without loss of generality, that $u = v_1$. Since G has no adjacent 4-cycles, w is not adjacent to v_2 nor v_k. By Lemma 2.4, w is not adjacent to any vertex of v_3, v_4, v_5 and v_6. By (1), $G[\{u, w\}] \in \mathscr{G}$. By the minimality of (G, C_0), $(G[\{u, v\}], C_0)$ is superextendable. By the definition of superextendability, the color of u is different from the colors of v and x. Thus, we color w with the color of u and get a desired $(2, 0, 0)$-coloring of G, a contradiction.

(3) Suppose otherwise that $f = [uvwx]$ is a $(4^-, 3^+, 4^-, 3^+)$-face of G. Let $H = G[\{v, x\}]$. As in the proof of (1), $H \in \mathscr{G}$. By the minimality of G, H is $(2, 0, 0)$-colorable. We now extend the $(2, 0, 0)$-coloring of H to a $(2, 0, 0)$-coloring of G. We color v and x with the color of v_{vx}, and keep the colors of the other vertices of H. The $(2, 0, 0)$-coloring of H cannot be extended to a $(2, 0, 0)$-coloring of G if and only if each of v_{vx}, u (or w) and one neighbor of u (or w) are colored with 1 in H. We assume, without loss of generality, that each of v_{vx}, u and one neighbor of u are colored 1. In this case, we can properly recolor u. So, we obtain a desired a $(2, 0, 0)$-coloring of G, a contradiction. ■

le6 **Lemma 2.9** *Let $|C_0| = k$, where $k = 3, 7, 9$.*

(1) *If f is a 3-face, then $|b(f) \cap C_0| \leq 2$. If $|C_0| = 3$, then $|b(f) \cap C_0| \leq 1$.*

(2) *Let f be a 4-face. If $b(f) \cap C_0 \neq \emptyset$, then $|b(f) \cap C_0| = 2$.*

Proof. (1) Since C_0 is an induced cycle, C_0 has no chord. Thus, $|b(f) \cap C_0| \leq 2$. If $|C_0| = 3$, then $|b(f) \cap C_0| \leq 1$ since G has no adjacent 3-cycles.

(2) Assume that $b(f) \cap C_0 \neq \emptyset$. Since C_0 is an induced cycle, $|b(f) \cap C_0| \leq 3$. By Lemma 2.8 (2), $|b(f) \cap C_0| \geq 2$. So we just need to show that $|b(f) \cap C_0| \neq 3$. Assume that $|b(f) \cap C_0| = 3$. Then f has three consecutive vertices on C_0, say v_1, v_2, v_3. Now v_1 and v_3 have a common neighbor in $int(C_0)$, so by Lemma 2.3, $v_1 v_3 \in E(G)$, which implies that we have a K_4^-, a contradiction. ■

We call a $(3, 4, 4)$-face or $(3, 3, 5^-)$-face in F_3 *light*. A 3-vertex is *light* if it is on a light 3-face.

le10 **Lemma 2.10** *(Lemma 3.10 [14]) Let $f = [uvw]$ be a light 3-face with $d(u) = 3$ and let $u' \notin C_0$ be a pendent neighbor of u. Then a $(2, 0, 0)$-coloring of $(G - \{u, u'\}, C_0)$ can be extended to the desired coloring of $G - u'$ such that u is colored with 1.*

Let $f = [uvw]$ be a $(3, 3, k)$-face such that $b(f) \cap C_0 = \emptyset$, where $k \geq 5$. A face f is *poor* or *semi-poor* if it has two or one pendent 4^--neighbors in $int(C_0)$ at u and v, respectively. It is called *rich* if it is not poor or semi-poor. Sometimes, we also say that f is *non-rich* if it is poor or semi-poor.

le15 **Lemma 2.11** *(Lemma 12 [6]) Let $f = [uvw]$ be a poor or semi-poor $(3,3,k)$-face of G with $d(w) = k$, and u' the pendent 4^--neighbor of u. If $G' = G - w$ has a $(2,0,0)$-coloring ϕ that is a superextension from C_0 to G'. Then G' also has a $(2,0,0)$-coloring ϕ_α that is also a superextension of ϕ from C_0 to G', such that $\phi_\alpha(x) = \phi(x)$ if $x \notin \{u', u, v\}$ and $\alpha \notin \{\phi_\alpha(u), \phi_\alpha(v)\}$), where $\alpha \in \{2,3\}$.*

Here we summarize some results obtained in [6] by applying Lemma 2.11.

le24 **Lemma 2.12** *Let v be a k-vertex in $int(C_0)$ with $k \geq 5$.*

(1) (Lemma 15 [6]) *if $k = 5$, then v cannot be incident with 4 light pendent 3-faces.*
(2) (Lemma 18 [6]) *If v is a 3-triangular 8-vertex and incident with three poor or semi-poor $(3,3,8)$-faces, then v is not incident with light 3-vertices.*
(3) (Lemma 17 [6]) *If a 9-vertex is incident with four $(3,3,9)$-faces, then at least one of them is rich.*
(4) (Lemma 16 [6]) *If a 10-vertex is incident with five $(3,3,10)$-faces, then at least one of them is rich.*

le13 **Lemma 2.13** *Let v be a 5-vertex with neighbors v_i, $0 \leq i \leq 4$, in a cyclic order. Then*

(1) *If v is incident with a $(3,4,5)$-face and adjacent to three pendent 3-faces, then it can be adjacent to at most one pendent light 3-face.*
(2) (Lemma 24(2), [6]) *Let v be a 1-triangular 5-vertex. If v is incident with a $(3,5^+,5)$-face, then it can be incident with at most two light 3-faces.*
(3) (Lemma 25, [6]) *Let $f_1 = [v_0v_1v]$ and $f_2 = [v_2v_3v]$ be two 3-faces. If both f_1 and f_2 are $(3,4^-,5)$-faces, then v_4 is a 4^+-vertex.*

Proof. (1) Suppose to the contrary that v is incident with a $(3,4,5)$-face $f = [v_0vv_1]$ with $d(v_1) = 4$ and adjacent to two pendent light 3-faces. We first assume that $[v_2x_1x_2]$ is a pendent light 3-face. In this case, $[v_3x_3x_4]$ or $[v_4x_5x_6]$ is a pendent 3-face. We prove the case that $[v_4x_5x_6]$ is a pendent light 3-face. The proof is similar for the case that $[v_3x_3x_4]$ is a pendent light 3-face. Denote by G' the graph obtained from G by deleting v, v_0, v_2, v_4 and identifying v_1 and v_3. By Lemma 2.6, $G' \in \mathscr{G}$. By the minimality of G, (G', C_0) is superextendable and G' has a $(2,0,0)$-coloring. We now go back to color the vertices of G. We keep the colors of all vertices of G'. By Lemma 2.10, we assign 1 to both v_2 with v_4. We assume first that $v_{v_1v_3}$ is colored with 1. We color both v_1 and v_3 with 1, and properly color v_0 and v. Thus, we get a $(2,0,0)$-coloring of G, a contradiction. Thus, by symmetry, assume that $v_{v_1v_3}$ is colored with 2. We properly color v_0. If v_0 is colored with 1, then properly color color v, a contradiction. Thus, assume that v_0 is colored with 3. In this case, let v_i' and v_i'' be the two neighbors of v_i rather than v_i for $i = 2, 4$. If both v_i' and v_i'' are both colored with 1, then recolor v_i properly for some $i \in \{2,4\}$. Otherwise, keep the color of v_i unchanged. Thus, we can color v with 1, a contradiction.

Thus, assume that $[v_2x_1x_2]$ is not a light face. In this case, let G' be the graph obtained from G by deleting v, v_0, v_2, v_3 and identifying v_1 and v_4. By Lemma 2.6, $G' \in \mathscr{G}$. By the minimality of G, (G', C_0) is superextendable and G' has a $(2,0,0)$-coloring. We now go back to color the vertices of G. We keep the colors of all vertices of G'.

We assume first that $v_{v_1v_4}$ is colored with 1. We color both v_1 and v_4 with 1. By Lemma 2.10, we assign 1 to v_3. We properly color v_0 and v_2. If both v_0 and v_2 are colored 2 or 3, then properly color v, a contradiction. Thus, assume that v_0 and v_2 are colored 2 and 3, respectively. Let v_1' and v_1'' be the neighbors of v rather than v and v_0 and let v_4' and v_4'' be the two neighbors of v_4 rather than v. If both v_1' and v_1'' are colored 1, then recolor v_1 with 3, and then color v with 1, a contradiction. Thus, assume that at most one of v_1' and v_1'' is colored with 1. In this case, we properly recolor v_3 and v_4. If at most one of v_3 and v_4 is colored with 1, then color v with 1, a contradiction. Thus, assume that both v_3 and v_4 are colored with 1. Let v_0' be the neighbor of v_0 rather than v and v_1. Note that v_0 is properly colored 2. If v_0' is colored 1, then recolor v_0 with 3 and then color v with 1; if v_0' is colored with 3, then recolor v_0 with 1, color v with 2. We get a contradiction in each case.

Next, we assume that $v_{v_1v_4}$ is colored with 2 by symmetry. We recolor v_3 properly. If each of v_0, v_2, v_3 is colored with 1, then color v with 3, a contradiction. If at most two of v_0, v_2, v_3 are colored with 1, then color v with 1, a contradiction. ■

le13a **Lemma 2.14** (Lemma 30, [6]) *Let v be a 5-vertex with neighbors v_i, $0 \le i \le 4$, in cyclic order. If $f_1 = [v_0v_1v]$ is a $(3,5^+,5)$-face, $f_2 = [v_2v_3v]$ is a $(3,4,5)$-face, and v_4 is a light 3-neighbor of v, then the pendent neighbor of the 3-vertex of f_2 is a 3^+-vertex on C_0 or a 5^+-vertex.*

le16 **Lemma 2.15** *Let v be a 6-vertex with neighbors v_i, $0 \le i \le 5$. Then each of the following holds.*

(1) *If v is a 1-triangular 6-vertex incident with one non-rich (3,3,6)-face, then it is incident with at most two pendent light 3-faces.*

(2) *v is incident with at most one non-rich $(3,3,6)$-face.*

Proof. (1) Let $f_1 = [v_0v_1v]$ be a $(3,3,6)$-face. Suppose otherwise that v is incident to three pendent light 3-faces $f_2 = [v_2v_2'v_2'']$, $f_3 = [v_3v_3'v_3'']$ and $f_4 = [v_4v_4'v_4'']$. By Lemma 2.10, $(G - \{v, v_2, v_3, v_4\}, C_0)$ has a $(2,0,0)$-coloring which can be extended to a $(2,0,0)$ of $G - v$ such that each of v_2, v_3 and v_4 is colored with 1. If v_5 is colored with 1, then we may assume that v_0 and v_1 are colored with 1 and 2, respectively, by Lemma 2.11. Thus, we can color v with 3, a contradiction. Next, we may assume that v_5 is colored with 2 by symmetry of 2 and 3. By Lemma 2.11, $G - v$ has a $(2,0,0)$-coloring such that each of v_0 and v_1 is not colored with 3. Thus, we can color v with 3 and obtain a desired $(2,0,0)$-coloring of G, a contradiction.

(2) Suppose otherwise that v is incident with two non-rich $(3,3,6)$-faces $f_1 = [v_0v_1v]$ and $f_2 = [v_2v_3v]$. Let G' be the graph obtained from G by deleting vertex v. By Lemma 2.6, $G' \in \mathscr{G}$. By the minimality of G, the $(2,0,0)$-coloring of C_0 can be superextended to G'. We claim that v_4 and v_5 are colored with 2 and 3, respectively. Suppose otherwise that by symmetry none of v_4 and v_5 is colored with 2. By Lemma 2.11, the coloring $(2,0,0)$ of C_0 can be superextended to G' such that none of v_0 and v_1 is colored with 2 and so neither of v_2 and v_3. In this case, v can be colored with 2, a contradiction. Thus, assume that v_4 is colored with 2 and v_5 is colored with 3. Now, applying Lemma 2.11 again, v_0 and v_1 colored with 1 and 3, respectively, and v_2 and v_3 are also colored with 1 and 3, respectively. In this case, at most two neighbors of v are colored with 1. Thus, we can color v with 1, a contradiction. ∎

le17 **Lemma 2.16** *If v is a 2-triangular 6-vertex with neighbors v_i, where $0 \le i \le 5$, and $f_1 = [v_0v_1v]$ is a $(3,3,6)$-face, then each of the following holds:*

(1) *(Lemma 28(1), [6]) If f_2 is a $(3,4,6)$-face or $(3,3,6)$-face, then at least one of the isolated neighbor of v is a 4^+-vertex.*

(2) *(Lemma 28(2), [6]) If f_2 is a $(3,5^+,6)$-face, then at most one of the isolated neighbor of v is a light 3-vertex.*

le180 **Lemma 2.17** *Let v be a 6-vertex with neighbors v_i, where $0 \le i \le 5$. If v is a 2-triangular or 3-triangular, then v is incident with at most one non-rich $(3,3,6)$-face.*

Proof. Suppose otherwise that v is incident with two non-rich $(3,3,6)$-faces $f_1 = [v_0v_1v]$ and $f_2 = [v_2v_3v]$. Denote by G' the graph obtained from G by deleting v, v_0, v_1, v_2 and v_3. By Lemma 2.6, $G' \in \mathscr{G}$. By minimality of G, G' is $(2,0,0)$-colorable. Now we go back to color the vertices of G. We only prove the case that v is a 2-triangular 6-vertex. The proof is similar for the case that v is a 3-triangular 6-vertex. If v_4 and v_5 are both colored with 1 or 2, then v_0 and v_1 can be colored with 1 and 2, respectively, and so can v_2 and v_3 by Lemma 2.11. In this case, v can be colored with 3, a contradiction. If v_4 and v_5 are both colored with 3, then v_0 and v_1 can be colored with 1 and 3, respectively, and so can v_2 and v_3 by Lemma 2.11. In this case, v can be colored with 2, a contradiction. If v_4 and v_5 are colored with 1 and 2, respectively, then v_0 and v_1 can be colored with 1 and 2, respectively, and so can v_2 and v_3 by Lemma 2.11. In this case, v can be colored with 3, a contradiction. If v_4 and v_5 are colored with 1 and 3, respectively, then v_0 and v_1 can be colored with 1 and 3, respectively, and so can v_2 and v_3 by Lemma 2.11. In this case, v can be colored with 2, a contradiction. If v_4 and v_5 are colored with 2 and 3, respectively, then v_0 and v_1 can be colored with 1 and 2, respectively, and so can v_2 and v_3 by Lemma 2.11. In this case, v can be nicely colored with 1, a contradiction. ∎

le18 **Lemma 2.18** *Let v be a 3-triangular 6-vertex with neighbors v_i, where $0 \le i \le 5$. Let $f_1 = [v_0v_1v]$ be a $(3,3,6)$-face, $f_2 = [v_2v_3v]$ and $f_3 = [v_4v_5v]$. Then each of the following holds.*

(1) *(Lemma 29(2), [6]) If f_2 is a $(3,3,6)$-face, then f_3 has no 3-vertex.*
(2) *(Lemma 29(1), [6]) At most one of f_2 and f_3 is a $(3,4^-,6)$-face.*
(3) *(Lemma 29(3), [6]) If f_2 is a $(3,4,6)$-face and f_3 has a 3-vertex, then either f_1 is rich or the outer neighbor of the 3-vertex of f_2 is either a 3^+-vertex on C_0 or a 5^+-vertex.*

le20 **Lemma 2.19** *Let v be a 7-vertex with neighbors v_i, $0 \le i \le 6$. Then*

(1) *If v is 2-triangular and incident with two non-rich $(3,3,7)$-faces, then at most one of the three isolated 3-vertices is a light 3-vertex.*
(2) *If v is 3-triangular, then v is incident with at most one non-rich $(3,3,7)$-face.*

Proof. (1) Let $f_1 = [v_0vv_1]$ and $f_2 = [v_2vv_3]$ be two non-rich $(3,3,7)$-faces. Suppose to the contrary that v_4 and v_5 are two light 3-vertices. Denote by G' the graph obtained from G by deleting v. By Lemma 2.6, $G' \in \mathcal{G}$. By the minimality of G, G' has a $(2,0,0)$-coloring. Now we extend the $(2,0,0)$-coloring of G' to a $(2,0,0)$-coloring of G. Assume first that v_6 is colored with 1 or 2. By Lemma 2.11, we can recolor v_0, v_1 with 1 or 2, respectively, and so can v_2 and v_3, respectively. By Lemma 2.10, we can recolor both v_4 and v_5 with 1. Thus, we can color v with 3, a contradiction. Thus, assume that v_6 are colored with 3. In this case, by Lemma 2.11, we can recolor v_0, v_1 with 1 and 3, respectively, and recolor v_2, v_3 with 1 and 3, respectively. By Lemma 2.10, we can recolor both v_4 and v_5 with 1. Thus, color v with 2, a contradiction.

(2) Let $f_1 = [v_0vv_1]$, $f_2 = [v_2vv_3]$ and $f_3 = [v_4vv_5]$. Suppose otherwise that two of f_1, f_2 and f_3 are non-rich $(3,3,7)$-faces. We only prove the case that f_1 and f_2 are two non-rich $(3,3,7)$-faces. The proof is similar for the cases that f_1 and f_3 are two non-rich $(3,3,7)$-faces and that f_2 and f_3 are two non-rich $(3,3,7)$-faces. Denote by G' the graph obtained from G by deleting v and then identifying v_4 and v_6. By Lemma 2.6, $G' \in \mathcal{G}$. By the minimality of G, G' has a $(2,0,0)$-coloring. We now go back to color the vertices of G. We color each of v_4 and v_6 with the color of $v_{v_4v_6}$.

We first assume that $v_{v_4v_6}$ is colored with 1. If v_5 is colored with 1 or 2, then we can color v_0 and v_1 with 1 and 2, respectively, and recolor v_2 and v_3 with 1 and 2, respectively, by Lemma 2.11. In this case, we color v with 3, a contradiction. If v_5 is colored with 3, then by Lemma 2.11, we can color v_0 and v_1 with 1 and 3, respectively, and recolor v_2 and v_3 with 1 and 3, respectively. In this case, we color v with 2, a contradiction. Thus, we assume that $v_{v_4v_6}$ is colored with 2 by symmetry of 2 and 3. In this case, v_5 cannot be colored with 2. If v_5 is colored with 3, then by by Lemma 2.11, we can color v_0 and v_1 with 1 and 3, respectively, and recolor v_2 and v_3 with 1 and 3, respectively. In this case, we can nicely color v with 1, a contradiction. Thus, assume that v_5 is colored with 1. By Lemma 2.11, we can color v_0 and v_1 with 1 and 2, respectively, and recolor v_2 and v_3 with 1 and 2, respectively. In this case, we color v with 3, a contradiction. ■

le22 **Lemma 2.20** *(Lemma 26 (1), [6]) Let v be a 4-triangular 8-vertex with neighbors v_i, $0 \le i \le 7$. If all incident 3-faces of v are $(3,3,8)$-faces, then v is incident with at most two non-rich $(3,3,8)$-faces.*

3 Discharging procedure

In this section, we will finish the proof of Theorem 1.6 by a discharging argument. Let the initial charge of vertex $u \in G$ be $\mu(u) = 2d(u) - 6$, and the initial charge of face $f \ne C_0$ be $\mu(f) = d(f) - 6$, and $\mu(C_0) = d(C_0) + 6$. Then

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = 0.$$

We first give some more definitions here. A 5-vertex v is *bad* if it is incident with a $(3,4,5)$-face and a $(3,5,5^+)$-face, and the isolated neighbor of v is a light 3-vertex. A 6-vertex v is *bad* if it is incident with a $(3,3,6)$-face, a $(3,4,6)$-face and a $(3,5^+,6)$-face. The discharging rules are as follows.

(R1) Let u be a vertex not on C_0.

(R1.1) Every 4-vertex u gives 1 to each incident 3-face.

(R1.2) Every 5-vertex u gives $\frac{3}{2}$ to the incident $(3,5,5^+)$-face and 1 to the incident $(3,3,5)$-face and $(4^+,4^+,5)$-face. Moreover, u gives 1 to each light pendent 3-faces and $\frac{1}{2}$ to each other pendent 3-faces. The vertex u gives $\frac{3}{2}$ to the incident $(3,4,5)$-face if it is bad and 2 to the incident $(3,4,5)$-face otherwise.

(R1.3) Every k-vertex u with $k \geq 6$ gives 2, $\frac{5}{2}$, and 3 to the incident $(3,3,k)$-face if it is rich, semi-rich and poor, respectively. The vertex u gives $\frac{3}{2}$ to the incident $(3,5^+,k)$-face, and 1 to other incident 3-face. Moreover, u gives 1 to each light pendent 3-face and $\frac{1}{2}$ to each other pendent 3-face. Every 7^+-vertex gives 2 to the incident $(3,4,k)$-face. Let u be a 6-vertex. If u is bad, u gives $\frac{3}{2}$ to the incident $(3,4,6)$-face if u is not incident with a rich $(3,3,6)$-face and 2 to the incident $(3,4,6)$-face otherwise. If u is not bad, it sends 2 to the incident $(3,4,6)$-face.

(R1.4) Every 5^+-vertex u gives 1 to each incident $(4^-,4^-,5^+,5^+)$-face, and gives $\frac{2}{3}$ to each incident $(4^-,5^+,5^+,5^+)$-face, and gives $\frac{1}{2}$ to each incident $(5^+,5^+,5^+,5^+)$-face.

R2 (R2) C_0 gives 3 to each face in $F_3' \cup F_3''$, 2 to each face in F_4'', and 1 to each pendent 3-face.

R3 (R3) Every 6^+-face f $(f \neq C_0)$ sends $d(f) - 6$ to C_0 and every vertex $u \in C_0$ gives $2d(u) - 6$ to C_0.

We shall show that each $x \in V(G) \cup F(G) \setminus \{C_0\}$ has final charge $\mu^*(x) \geq 0$, and $\mu^*(C_0) > 0$.

First we consider faces. By (R3), for each 6^+-face f $(f \neq C_0)$, $\mu^*(f) = 0$. Since G contains no 5-faces, we first consider 3-faces and 4-faces other than C_0. Let f be a 3- or 4-face. By Lemma 2.9, $|b(f) \cap C_0| \leq 2$. If $|b(f) \cap C_0| \geq 1$, then by (R2), $\mu^*(f) = 0$. Thus, we may assume that $b(f) \cap C_0 = \emptyset$.

If f is a 4-face in F_4, then by Lemma 2.8 (3), f contains at least two 5^+-vertices, then f gains 2 from the 5^+-vertices by (R1.4), so $\mu^*(f) = 0$.

Let f be a 3-face in F_3. Let $f = [uvw]$ with degree sequence (d_1, d_2, d_3).

(1) f is a $(3,3,5^-)$-face. If f is a $(3,3,5)$-face or $(3,3,4)$-face, then by Lemma 2.1(5) (6), each of the pendent neighbor of f is either a 5^+-vertex or on C_0. Then f gets 1 from each of the pendent neighbor of f by (R1.2.1) and (R1.3.1), and at least 1 from the incident 4^+-vertex by (R1.1), and gets 1 from C_0 by (R2). Thus, $\mu^*(f) \geq 3 - 6 + 1 \times 3 = 0$. If f is a $(3,3,3)$-face, then by Lemma 2.1(5) and (6), each of the pendent neighbor of f is either a 5^+-vertex or on C_0. In this case, f gets 1 either from each of the pendent neighbor of f by (R1.2.1) and (R1.3.1) or from C_0 by (R2). Thus, $\mu^*(f) \geq 3 - 6 + 1 \times 3 = 0$.

(2) f is a $(3,3,6^+)$-face. By (R1.3.1), f receives 2 or $\frac{5}{2}$ or 3 from w if it is rich or semi-rich or poor. If a pendent neighbor of a 3-vertex is on C_0, then C_0 gives 1 to f by (R2). This implies that $\mu^*(f) \geq 3 - 6 + 1 + 2 = 0$. Thus, we assume that no pendent neighbor of 3-vertex is on C_0. If f is poor, then w sends 3 to f. Thus, $\mu^*(f) \geq 3 - 6 + 3 = 0$. If f is semi-rich, then there exists exactly one pendent neighbor of a 3-vertex is a 5^+-vertex, which sends $\frac{1}{2}$ to f by (R1.2.1) and (R1.3.1), also w sends $\frac{5}{2}$ to f by (R1.3.1). Thus, $\mu^*(f) \geq 3 - 6 + \frac{5}{2} + \frac{1}{2} = 0$. Now we assume that f is rich, each pendent neighbor of the two 3-vertices is a 5^+-vertex. By (R1.2.1) and (R1.3.1), each of them gives $\frac{1}{2}$ to f, also w sends 2 to f by (R1.3.1). Thus, $\mu^*(f) \geq 3 - 6 + 2 + 2 \times \frac{1}{2} = 0$.

(3) f is a $(3,4,4)$-face. By Lemma 2.1(5)(6), the pendent neighbor u' of 3-vertex u is either on C_0 or is a 5^+-vertex. In the former case, u' gives 1 to f by (R2). By (R1.1), each of v and w sends at least 1 to f. Thus, $\mu^*(f) \geq 3 - 6 + 1 \times 3 = 0$. In latter case, f is a light pendent 3-face. By (R1.2.2) and (R1.3.1), u' sends 1 to f. By (R1.1), f gets 1 from v and w respectively. Thus, $\mu^*(f) \geq 3 - 6 + 1 \times 3 = 0$.

(4) f is a $(3,4,5)$-face. If w is not a bad vertex, then by (R1.1), (R1.2.1), v sends 1 to f and w sends 2 to f. Thus, $\mu^*(f) \geq 3 - 6 + 1 + 2 = 0$. Thus, assume that w is bad. By Lemma 2.14, the pendent neighbor of u is a 3^+-vertex on C_0 or a 5^+-vertex z. In the former case, by (R1.1), (R1.2.1) and (R2), f gets 1 from v, gets $\frac{3}{2}$ from w and gets 1 from the 3^+-vertex on C_0. Thus, $\mu^*(f) \geq 3 - 6 + 1 + \frac{3}{2} + 1 > 0$. In the latter case, by (R1.1), (R1.2.1), f gets 1 from v, gets $\frac{3}{2}$ from w and gets $\frac{1}{2}$ from the 5^+-vertex which is the pendent neighbor of u. Thus, $\mu^*(f) \geq 3 - 6 + 1 + \frac{3}{2} + \frac{1}{2} = 0$.

(5) f is a $(3,4,6)$-face. If w is not a bad vertex, then by (R1.1) and (R1.3.1), v sends 1 to f and w sends 2 to f. Thus, $\mu^*(f) \geq 3-6+1+2=0$. Thus, assume that w is bad. By Lemma 2.18, either w is incident with a rich $(3,3,6)$-face or the pendent neighbor of u is a 3^+-vertex on C_0 or a 5^+-vertex. If w is incident with a rich $(3,3,6)$-face, then w gives 2 to f by (R1.3.1) and v sends 1 to f by (R1.1). Thus, $\mu^*(f) \geq 3-6+1+2=0$. If the pendent neighbor of u is a 3^+-vertex on C_0, by (R1.1), (R1.3.1) and (R2), f gets 1 from v, gets $\frac{3}{2}$ from w and gets 1 from the 3^+-vertex on C_0. Thus, $\mu^*(f) \geq 3-6+1+\frac{3}{2}+1>0$. If the pendent neighbor of u is a 5^+-vertex, then f gets 1 from v, gets $\frac{3}{2}$ from w and gets $\frac{1}{2}$ from the 5^+-vertex which is the pendent neighbor of u by (R1.1), (R1.3.1). Thus, $\mu^*(f) \geq 3-6+1+\frac{3}{2}+\frac{1}{2}=0$.

(6) f is a $(3,4,7^+)$-face. By (R1.1), (R1.3.1), v sends 1 to f and w sends 2 to f. Thus, $\mu^*(f) \geq -3+1+2=0$.

(7) f is a $(3,5^+,5^+)$-face. By (R1.2.1) and (R1.3.1), each of v and w sends $\frac{3}{2}$ to f. Thus, $\mu^*(f) \geq -3+2\times\frac{3}{2}=0$.

(8) f is a $(4^+,4^+,4^+)$-face. By (R1.1), (R1.2.1) and (R1.3.1), f gets at least 1 from each of u, v and w. Thus, $\mu^*(f) \geq -3+1\times 3=0$.

Now we consider vertices. By (R3), for each vertex $u \in C_0$, $\mu^*(u) = 2d(u)-6-(2d(u)-6)=0$. So we only need to consider vertices in $int(C_0)$. By Lemma 2.1, $int(C_0)$ contains no 2^--vertices. For $u \notin C_0$, let p,q,t be the number of incident 4-faces pendent 3-faces, and incident 3-faces of u, respectively. Let t' be the number of rich $(3,3,d(u))$-faces and $(3,4^+,d(u))$-faces and let q' be the number of non-light pendent 3-faces and neighbors not on 3-faces. Since G contains no, 5-cycle, K_4^-, or adjacent 4-faces, we have

$$2p+q+2t \leq d(u). \tag{1}$$

If $d(u)=3$, by the discharging rules $\mu^*(u)=\mu(u)=0$. Thus, we consider $d(u)\geq 4$.

Lemma 3.1 *Every 7^+-vertex in $int(C_0)$ has nonnegative final charge.*

Proof. Let $u \in int(C_0)$ with $d(u)=k\geq 7$. By (R1.3), we have

$$\begin{aligned}\mu^*(u) &\geq 2d(u)-6-(p+q+3t-t'-\frac{1}{2}q') = 2d(u)-6-(2p+q+2t)-t+t'+p+\frac{1}{2}q' \\ &\geq d(u)-6-t+t'+p+\frac{1}{2}q' \geq d(u)-6-\lfloor\frac{d(u)}{2}\rfloor+t'+p+\frac{1}{2}q' = \lceil\frac{d(u)}{2}\rceil-6+t'+p+\frac{1}{2}q'.\end{aligned}$$

So $\mu^*(u)\geq 0$ if $d(u)\geq 11$. If $d(u)\in\{9,10\}$, then by Lemma 2.12 (3) and (4), u is incident with a rich $(3,3,k)$-face or a $(3^+,4^+,k)$-face, that is, $t'\geq 1$. So $\mu^*(u)\geq 5-6+1=0$. Now let $d(u)=8$. Then by Lemma 2.20 and Lemma 2.12 (2), $t\leq 2$, or $t=4$ and $t'\geq 2$, or $t=3$ and $q'=2$. In either case, $\mu^*(u)\geq 8-6-t+t'+\frac{1}{2}q'\geq 0$. Let $d(u)=7$. By Lemma 2.19, $t=3$ and $t'\geq 2$, or $t=2$ and $q'\geq 2$, or $t\leq 1$. In either case, $\mu^*(u)\geq 7-6-t+t'+\frac{1}{2}q'\geq 0$. ■

Lemma 3.2 *Each 4-vertex has nonnegative final charge.*

Proof. Let u be a 4-vertex. Since G has no 5-cycle, u is incident with at most two 3-faces. If u is incident with two 3-faces, then by (R1.1), u gives 1 to each incident 3-face and $\mu^*(u)=2-1\times 2=0$. If u is incident with only one 3-face, then u is incident with at most one 4-face since G has no 5-cycle. Thus, by (R1.1), u gives 1 to the incident 3-face. This implies that $\mu^*(u)\geq 2-1=1>0$. If u is not incident with any 3-face, then by (R1.1), $\mu^*(u)\geq 2>0$. ■

Lemma 3.3 *Each 5-vertex has nonnegative final charge.*

Proof. Let u be a 5-vertex. Let u be not a bad vertex. Assume first that u is not incident with any 3-faces. Since G has no adjacent two 4-faces, u is incident with at most two 4-faces. If u is incident with two 4-faces, then u is incident with at most two $(4^-,4^-,5,5^+)$-faces. In this case, u is adjacent to at most one pendent 3-faces. Thus, $\mu^*(u)\geq 4-2-1\geq 0$ by (R1.2.1). If u is incident with one 4-faces, then u

is adjacent to at most three pendent light 3-faces. Thus, $\mu^*(u) \geq 4-1-3=0$ by (R1.2.1). If u is not incident with any 4-face, u can be incident with at most three light pendent 3-faces by Lemma 2.12(1). By (R1.2.1), u gives 1 to each of these three light pendent and $\frac{1}{2}$ to the other two pendent 3-faces. Thus, $\mu^*(u) \geq 4-1\times 3-2\times\frac{1}{2}=0$.

Thus, we assume that u is incident with at least one 3-face f_1. Consider that u is 1-triangular. Since G has no adjacent two 4-faces, u is at most one 4-face. In this case, u is either incident with a 4-face and at most one pendent 3-face or at most three pendent 3-faces. In the former case, u gives at most 2 to the incident 3-face, gives 1 to the incident 4-face and at most 1 to the pendent 3-face by (R1.2.1) and (R1.2.2). Thus, $\mu^*(u) \geq 4-2-1-1=0$. In the latter case, If f_1 is a $(3,3,5)$-face, then u gives 1 to f_1, at most 1 to each pendent 3-face by (R1.2.1) and (R1.2.2). Thus, $\mu^*(u) \geq 4-1\times 4=0$. If f_1 is a $(3,4,5)$-face, then u is adjacent to one pendent light 3-face by Lemma 2.13(1). By (R1.2.1), u gives at most 2 to the incident 3-face, at most 1 to the light pendent 3-face and $\frac{1}{2}$ to each other pendent 3-face. Thus, $\mu^*(u) \geq 4-2-1-\frac{1}{2}\times 2=0$. If f_1 is a $(3,5,5^+)$-face, then u is adjacent to at most two light 3-vertex by Lemma 2.13(3). By R(1.2.1), u gives $\frac{3}{2}$ to the incident $(3,5,5^+)$-face, 1 to each light pendent 3-face and $\frac{1}{2}$ to other the pendent 3-face. Thus, $\mu^*(u) \geq 4-\frac{3}{2}-2\times 1-\frac{1}{2}=0$. If f_1 is a $(4^+,4^+,5)$-face, then u is adjacent to at most three pendent 3-faces. In this case, u gives 1 to the incident $(4^+,4^+,5)$-face and gives at most 1 to each pendent 3-face. Thus, by (R1.2.1), $\mu^*(u) \geq 4-1-1\times 3=0$.

Now, we assume that u is 2-triangular, let f_1 and f_2 be the two 3-faces incident with u. If both of f_1 and f_2 are $(3,4^-,5)$-faces, then the isolated neighbor is a 4^+-neighbor by Lemma 2.13(3), hence $\mu^*(u) \geq 4-2\times 2=0$ by (R1.2.1). If none of f_1 and f_2 is a $(3,4^-,5)$-face, then u is adjacent to a pendent 3-face. Thus, $\mu^*(u) \geq 4-2\times\frac{3}{2}-1=0$ by (R1.2.1). Thus, assume that f_1 is a $(3,4^-,5)$-face and f_2 is a $(3,5,5^+)$-face. If f_1 is a $(3,3,5)$-face, then by (R1.2), u gives 1 to f_1 and gives $\frac{3}{2}$ to f_2. Thus, $\mu^*(u) \geq 4-1-\frac{3}{2}-1=\frac{1}{2}$. Assume that f_1 is a $(3,4,5)$-face. If u is not a bad 5-vertex, then the isolated neighbor is not a light 3-neighbor. In this case, $\mu^*(u) \geq 4-2-\frac{3}{2}-\frac{1}{2}=0$ by (R1.2.1). If u be a bad vertex, then the isolated neighbor is a light 3-neighbor. By (R1.2.1), u gives $\frac{3}{2}$ to $(3,5,5^+)$-face, $\frac{3}{2}$ to $(3,4,5)$-face and 1 to the light pendent 3-face. Thus, $\mu^*(u) \geq 4-2\times\frac{3}{2}-1=0$. ■

Lemma 3.4 *Each 6-vertex has nonnegative final charge.*

Proof. Let u be a 6-vertex with neighbor v_i, where $0 \leq i \leq 5$. Assume first that u is not a bad vertex. If u is not incident with any 3-faces, then $p+q \leq 6$. By (R1.3), u gives at most 1 to each of the pendent 3-faces or incident 4-faces. Thus, $\mu^*(u) \geq 6-1\times 6=0$. If u is 1-triangular with $f_1=[v_0v_1u]$, then $p+q \leq 4$. If f_1 is a rich $(3,3,6)$-face or a $(3,4^+,6)$-face, then u gives at most 2 to the incident 3-face. By (R1.3.1), $\mu^*(u) \geq 6-2-1\times 4=0$. If f_1 is a non-rich $(3,3,6)$-face, then by Lemma 2.12(2) at most two of the isolated neighbors of u are light 3-vertices. Thus, $\mu^*(u) \geq 6-3-1\times 2-\frac{1}{2}\times 2=0$ by (R1.3.1).

If u is 2-triangular, then $p=1$ or $q \leq 2$. Let $f_1=[v_0v_1u]$ and $f_2=[v_2v_3u]$ be the two 3-faces incident with u. In the case that $p=1$, let f_3 is a 4-face incident with u. By Lemma 2.15(2), at most one of f_1 and f_2 is a non-rich $(3,3,6)$-face. By (R1.3.1) and (R1.3.2), u gives at most 1 to each incident 4-face, at most 3 to the incident non-rich 3-face $(3,3,6)$-face and at most 2 the other 3-face. Thus, $\mu^*(u) \geq 6-3-2-1=0$. Thus, assume that $q \leq 2$. By Lemma 2.15(2), at most one of f_1 and f_2 is non-rich. Assume first that both f_1 and f_2 are rich. In this case, u gives 2 to each of the incident rich $(3,3,6)$-face and at most 1 to each of the pendent 3-face by R(1.3.1). Thus, $\mu^*(u) \geq 6-2\times 2-2\times 1=0$. Thus, assume that f_1 is non-rich $(3,3,6)$-face and f_2 is rich. If f_2 is a $(3,3,6)$-face or $(3,4,6)$-face, then at least one of v_4 and v_5 is a 4^+-vertex by Lemma 2.16(1). This means that u is adjacent to at most a pendent 3-face. In this case, u gives at most 3 to f_1, 2 to f_2 and at most 1 to the pendent 3-face. Thus, $\mu^*(u) \geq 6-2-3-1=0$. If f_2 is a $(3,5^+,6)$-face, then at most one of v_4 and v_5 is a light 3-vertex by Lemma 2.16(2). By (R1.3.1), u gives 3 to f_1 and $\frac{3}{2}$ to f_2 and 1 to the light pendent 3-face and gives $\frac{1}{2}$ to each non-light pendent 3-face. Thus, $\mu^*(u) \geq 6-3-\frac{3}{2}-1-\frac{1}{2}=0$. If f_2 is a $(4^+,4^+,6)$-face, then u gives 3 to f_1 and 1 to f_2 and at most 1 to each pendent 3-face (if they exist). Thus, $\mu^*(u) \geq 6-3-1-2\times 2=0$.

If v is a 3-triangular 6-vertex, let f_1, f_2 and f_3 be three incident 3-faces incident with u. By Lemma 2.17, u is incident with at most one non-rich 3-face. Assume first assume that none of f_1, f_2 and f_3 is a non-rich 3-face. By (R1.3.1), u gives at most 2 to each of f_1, f_2 and f_3. Thus, $\mu^*(u) \geq 6-2\times 3=0$. Thus, assume that f_1 is a non-rich $(3,3,6)$-face. If f_2 is a $(3,3,6)$-face, then f_2 is rich. By Lemma 2.18(1), f_3 has no 3-vertex. Thus, u gives 3 to f_1, 2 to f_2 and 1 to f_3 by (R1.3.1). So, $\mu^*(u) \geq 6-3-2-1=0$. If f_2 is a $(3,4,6)$-face, then by Lemma 2.18 (2), f_3 is not a $(3,4,6)$-face. Since u is not a bad vertex, f_3 has no 3-vertex. By (R1.3.1), u gives 3 to f_1, gives 2 to f_2, and gives 1 to f_3. Thus, $\mu^*(u) \geq 6-3-2-1=0$. If f_2 is a $(3,5^+,6)$-face, then we may assume that f_3 is a $(3,5^+,6)$-face or a $(4^+,4^+,6)$-face by argument above. In this case, u gives 3 to f_1, and gives at most $\frac{3}{2}$ to each of f_2 and f_3 by (R1.3.1). Thus, $\mu^*(u) \geq 6-3-2\times\frac{3}{2}=0$. Finally, we assume that both f_2 and f_3 are $(4^+,4^+,6)$-faces. By (R1.3.1), u gives 3 to f_1 and 1 to each of f_2 and f_3. Thus, $\mu^*(u) \geq 6-3-1\times 2>0$.

Let u be a bad vertex. Then u is incident with a $(3,3,6)$-face, a $(3,4,6)$-face and a $(3,5^+,6)$-face. By (R1.3.1), u gives $\frac{3}{2}$ to the $(3,5,6)$-face, $\frac{3}{2}$ to the $(3,4,6)$-face and 3 to the $(3,3,6)$-face. Thus, $\mu^*(u) \geq 6-2\times\frac{3}{2}-3=0$. ■

Now we consider the final charge of C_0. Assume that f_p is the number of 3-vertices adjacent to the vertices of C_0. Let x be the charge that C_0 gets from other 6^+-face by (R4). By (R2), (R3) and (R4),

$$\begin{aligned}\mu^*(C_0) &\geq d(C_0)+6+\sum_{u\in V(C_0)}(2d(u)-6)-3(|F_3'|+|F_3''|)-2|F_4''|-f_p+x\\ &= d(C_0)+6+2\sum_{u\in V(C_0)}(d(u)-2)-2|C_0|-3(|F_3'|+|F_3''|)-2|F_4''|-f_p+x\\ &= 6-|C_0|+2e(C_0,V(G)-V(C_0))-3(|F_3'|+|F_3''|)-2|F_4''|-f_p+x\\ &= 6-|C_0|+|F_3'|+|F_3''|+2|F_4''|+f_p+x+2e'. \qquad (2)\end{aligned}$$

where $e(C_0,V(G)-C_0)$ is the number of edges between C_0 and $V(G)-C_0$ and e' is the number of edges in $(C_0,V(G)-C_0)$ which are neither on any 3-face nor adjacent to an internal 3-vertex. The last equality above holds since each face from $F_3'\cup F_3''\cup F_4''$ counts two times in $e(C_0,V(G)-C_0)$ and each 3-neighbor of C_0 counts once in $e(C_0,V(G)-C_0)$.

In order to show that $\mu^*(C_0)>0$, it is sufficient for us to prove that $6-|C_0|+|F_3'|+|F_3''|+2|F_4''|+f_p+x+2e'>0$. If $|C_0|=3$, then it holds. Thus, we need to prove that if $|C_0|\in\{7,9\}$, then the inequality holds. Suppose otherwise that for $|C_0|\in\{7,9\}$,

$$6-|C_0|+|F_3'|+|F_3''|+2|F_4''|+f_p+x+2e'\leq 0. \qquad (3)$$

Assume first that $|C_0|=7$. From (3), we obtain that $|F_3'|+|F_3''|+2|F_4''|+f_p+x+2e'\leq 1$. This implies that $|F_4''|=e'=0$ and at most one of $|F_3'|,|F_3''|$ and f_p is 1. If one of $|F_3'|,|F_3''|$ and f_p is 1, then C_0 contains at least five 2-vertices, thus $x\geq 1$, contrary to (3). Thus, $|F_3'|=|F_3''|=f_p=0$ and C_0 is a cycle with seven 2-vertices, a contradiction.

Finally, we assume that $|C_0|=9$. From now on, we assume that $C_0=v_1v_2\dots v_9$.

Claim 1. $|F_4''|=0$ and $e'=0$.

Proof of Claim 1. Suppose first that $|F_4''|\neq 0$. By (3), $|F_4''|=1$ and $|F_3'|+|F_3''|+f_p+e'+x\leq 1$. Thus, C_0 has at least seven 2-vertices and hence G has a 7^+-face rather than C_0, which implies that $x\geq 1$. But then $|F_3'|=|F_3''|=f_p=e'=0$, which implies that C_0 has nine 2-vertices, and thus $x\geq 2$, a contradiction.

Suppose now that $e'\neq 0$. By (3), $|e'|=1$, $|F_4''|=0$ and $|F_3'|+|F_3''|+f_p+e'+x\leq 1$. It follows that C_0 contains at least six 2-vertices. By the definition of e', the edge that is counted in e' is not incident to any triangle, so it must be in 8^+-face, which implies $x\geq 2$, a contradiction. Thus we have Claim 1.

Let $U=\{v\in V(C_0): d(v)\geq 3\}=\{v_{i_1},v_{i_2},\dots,v_{i_t}\}$ with $i_1<i_2<\dots<i_t$ so that the vertices in U appear on C_0 in clockwise order. Let M_j be the path from v_{i_j} to $v_{i_{j+1}}$ and M_t be the path from v_{i_t} to v_{i_1} following the clockwise order. Let m_i be the number of interior vertices on M_i. Without loss of generality, we assume that $m_1=\max_{1\leq i\leq t} m_i$. Note that $t=2|F_3''|+|F_3'|+f_p$ and $\sum_{i=1}^t m_i=9-t$.

For simplicity, we assume that $v_{i_1}=v_1$. Let f_i denote the internal face whose boundary contains M_i. Note that C_0 has no chord. By Lemmas 2.3 and 2.4, the f_i must contain a path of length at least 4 whose vertices are all in $int(C_0)$ between v_{i_j} and $v_{i_{j+1}}$. Thus we obtain the following claim.

Claim 2. If $m_j\geq 1$ for some $j\in\{1,\dots,t\}$, then f_j is a $(m_j+5)^+$-face.

Let $t' = |\{m_j : m_j > 0\}|$. By (3) and Claim 1, $|F_3'| + |F_3''| + f_p + x \leq 3$ and thus $t' \leq 3$. If $|F_3'| + |F_3''| + f_p = 3$, then $\sum_{i=1}^{t} m_i \geq 3$. If $\sum_{i=1}^{t} m_i \geq 4$, then $m_1 \geq 2$. By Claim 2, G has a 7^+-face, hence $x \geq 1$, contrary to (3). Thus, $\sum_{i=1}^{t} m_i = 3$ and $|F_3''| = 3$ and $|F_3'| = f_p = x = 0$. Thus, we may assume that $[v_1v_2x_1], [v_4v_5x_2]$ and $[v_7v_8x_3]$ are three 3-faces from F_3'' and there is a 3-vertex y adjacent to each of x_1, x_2 and x_3, contrary to Lemma 2.1(1). If $|F_3'| + |F_3''| + f_p = 2$, $\sum_{i=1}^{t} m_i \geq 5$. Note that $t' \leq 2$. Thus, $m_1 \geq 3$. By Claim 2, G has a 8^+-face. By (R4), $x \geq 2$, contrary to (3). If $|F_3'| + |F_3''| + f_p = 1$, $\sum_{i=1}^{t} m_i \geq 7$. Note that $t' = 1$. Thus, $m_1 \geq 7$. By Claim 2, G has a 9^+-face. By (R3), $x \geq 3$, contrary to (3). Thus, $|F_3'| + |F_3''| + f_p = 0$. In this case, G is a 9-cycle, a contradiction.

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