# Matrix Problems in Quantum Information Science

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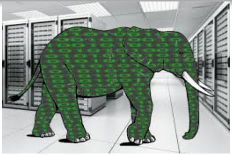

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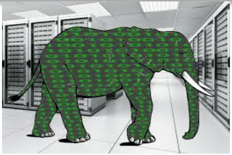

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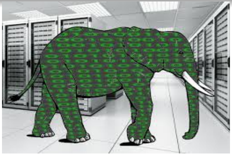

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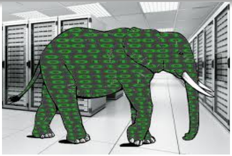

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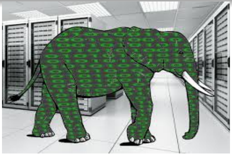

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What is considered "big data" varies depending on the capabilities of the users and their tools, and expanding capabilities make big data a moving target.

- To study big data, one may focus on analyzing important data sets, and deduce useful information and decisions.
- Alternatively, one may focus on some learning and creating techniques in handling large data set.

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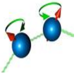
- That is why Richard Feynman suggested the use of quantum properties/systems to do fast computing.

- In quantum information science, one uses quantum states and properties to store, transmit, and process information.



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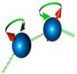

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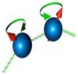

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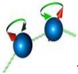

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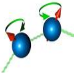

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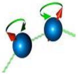

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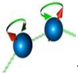

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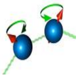

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*Quantum Operations*

**A Basic Problem**

Given two quantum states $\rho_1, \rho_2$ and a certain quantum operation or channel $\Phi$, how similar and how different can $\rho_1$ and $\Phi(\rho_2)$ be?

# Mathematical Formulation

- Quantum states are represented as $n \times n$ density matrices, i.e., positive semi-definite matrices with trace one, say, $\rho = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

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- Mixed unitary channel: $\Phi(X) = \sum_{j=1}^{r} p_j U_j X U_j^*$ for some unitary $U_1, \dots, U_r$ and probability vector $(p_1, \dots, p_r)$.

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- Unital channels: A quantum channel $\Phi$ such that $\Phi(I/n) = I/n$.

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- All quantum channels
  $\Phi(X) = \sum_{j=1}^{r} F_j X F_j^*$.

# Distance measures for quantum states

- For two numbers $a, b$, we can measure the distance between them by $|a - b|$.
- For two matrices / quantum states $\rho_1, \rho_2$, we can measure the distance between them by a norm
- There are different kinds of norms on matrices. For example, the operator norm $\|X\|_{oper} = \max\{\|Xv\| : v \in \mathbb{C}^n, \|v\| = 1\}$,

  the trace norm $\|X\|_1 = \mathrm{tr}\,|X|$, and Frobenius norm $\|X\|_F = \mathrm{tr}\,(X^*X)^{1/2}$.
- A norm $\|\cdot\|$ on $M_n$ is unitary similarity invariant (USI) if

$$\|UXU^*\| = \|X\| \text{ for any } U, X \in M_n \text{ such that } U \text{ is unitary.}$$

# Unitary channels: $\Phi(X) = UXU^*$

**Theorem [Li,Pelejo,Wang]**

Let $\|\cdot\|$ be a USI norm, $\rho_1 = U\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}U^*, \rho_2 \in M_n$ be density matrices, where $U$ is unitary and $a_1 \geq \cdots \geq a_n$. Suppose $\rho_2$ has eigenvalues $b_1 \geq \cdots \geq b_n$.

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3. There exists a unital quantum channel $\Phi$ such that $\Phi(\rho) = \sigma$.
4. $\lambda(\sigma) \prec \lambda(\rho)$, i.e., the sum of the $k$ largest eigenvalues of $\sigma$ is not larger than that of $\rho$ for $k = 1, \ldots, n-1$.

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  where $(d_1, \ldots, d_n)$ is determined by the following algorithm:

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  Step 1. If $\Delta_1 \geq \cdots \geq \Delta_n$, then set $(d_1, \ldots, d_n) = \lambda(\rho_1) - (\Delta_1, \ldots, \Delta_n)$ and stop.
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  Step 2. Let $2 \leq j < k \leq \ell \leq n$ be such that

  $$\Delta_{j-1} \neq \Delta_j = \cdots = \Delta_{k-1} < \Delta_k = \cdots = \Delta_\ell \neq \Delta_{\ell+1}.$$

  Replace each $\Delta_j, \dots, \Delta_\ell$ by $(\Delta_j + \cdots + \Delta_\ell)/(\ell - j + 1)$, and go to Step 1.

# Examples

Here are two examples illustrating the algorithm in the theorem.

**Example 1** Let $\rho_1 = \frac{1}{10}\text{diag}\,(4,3,3,0)$ and $\rho_2 = \frac{1}{10}\text{diag}\,(3,3,3,1)$.

**Apply Step 0:**

**Set** $(\Delta_1, \ldots, \Delta_4) = \frac{1}{10}\text{diag}\,(4,3,3,0) - \frac{1}{10}\text{diag}\,(3,3,3,1) = \frac{1}{10}\text{diag}\,(1,0,0,-1)$.

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**Example 1** Let $\rho_1 = \frac{1}{10}\text{diag}\,(4,3,3,0)$ and $\rho_2 = \frac{1}{10}\text{diag}\,(3,3,3,1)$.

**Apply Step 0:**

**Set** $(\Delta_1,\ldots,\Delta_4) = \frac{1}{10}\text{diag}\,(4,3,3,0) - \frac{1}{10}\text{diag}\,(3,3,3,1) = \frac{1}{10}\text{diag}\,(1,0,0,-1)$.

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- Our paper will be submitted and posted on arXiv soon.

## Thank you for your attention!

**Talk to me now or later if you have any questions!**

**Also talk to other EXTREEMS-QED faculty members if you are interested in their areas.**