Section 1.1 Systems of Linear Equations

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System of linear equations

- A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

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Ex: The following is a system of linear equations

$$2x_1 + 3x_2 - 4x_3 = 5 \quad (1)$$
$$5x_1 - 3x_2 - x_3 = 2 \quad (2)$$
$$3x_1 - 4x_2 + x_3 = -4 \quad (3)$$

- A solution of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

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- Two linear systems are called equivalent if they have the same solution set.

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Matrix Notation

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$$3x_1 - 4x_2 + x_3 = -4 \quad (6)$$

The matrix $\begin{pmatrix} 2 & 3 & -4 \\ 5 & -3 & -1 \\ 3 & -4 & 1 \end{pmatrix}$ is called the coefficient matrix of the system.

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- Here is the augmented matrix $\begin{pmatrix} 2 & 3 & -4 & 5 \\ 5 & -3 & -1 & 2 \\ 3 & -4 & 1 & -4 \end{pmatrix}$ of the system, which consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

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- The basic strategy for solving a linear system is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Solving System of Equations

- Example 1: solve the system of equations

$$x_1 - 2x_2 + x_3 = 0 \quad (7)$$
$$2x_2 - 8x_3 = 8 \quad (8)$$
$$-4x_1 + 5x_2 + 9x_3 = -9 \quad (9)$$

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- equations and the augmented matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix}$$

- Keep x_1 in (1) and eliminate it from the other equations (i.e. R3+4*R1)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{pmatrix}$$

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- Change the coefficient of x_2 in (2) to be 1 (i.e., R2*1/2)

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- Keep x_2 in (2) and eliminate it from (3) (i.e., R3+R2*3)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

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- What we have now is a matrix of triangular form.

- Now eliminate x_3 in (1) and (2) (that is, R2+R3*4, R1+R3*(-1))

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

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- Finally eliminate x_2 in (1): (that is, R1+R2*2)

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- To check the solution, we may just plug x_1, x_2, x_3 back to the original system of equations.

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- Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

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- Two fundamental questions about a linear system are as follows:
  - Is the system consistent; that is, does at least one solution exist? (existence)
  - If a solution exists, is it the only one; that is, is the solution unique?(uniqueness)

Existence and uniqueness of system of equations

- Example: determine if the following system is consistent

$$
\begin{aligned}
x_2 - 4x_3 &= 8 \\
2x_1 - 3x_2 + 2x_3 &= 1 \\
5x_1 - 8x_2 + 7x_3 &= 1
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Sol: The augmented matrix is $\begin{pmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{pmatrix}$

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- We can use row operations to get a triangular form (that is, interchange R1 and R2, R3+R1*(-5/2), R3+R2*(1/2))

$$\begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{pmatrix}$$

- We can write the corresponding system of equations

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\begin{aligned}
&2x_1 - 3x_2 + 2x_3 = 1\\
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- As the new system of equations and the original system of equations have the same solution set, the original system has no solution (i.e., is inconsistent).