# Section 1.2 Row Reduction and Echelon Forms

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# Echelon Form

- A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

$$\begin{pmatrix} 3 & -2 & 2 & 1 & 0 \\ 0 & 2 & 4 & -4 & 4 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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  - Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zeros.

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- An echelon matrix (respectively, reduced echelon matrix) is one that is in echelon form (respectively, reduced echelon form.)

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- However, the reduced echelon form one obtains from a matrix is unique.
  Theorem (Uniqueness of the Reduced Echelon Form) Each matrix is row equivalent to one and only one reduced echelon matrix.

# Pivot Postiion

- If a matrix $A$ is row equivalent to an echelon matrix $U$, we call $U$ an echelon form (or row echelon form) of $A$; if $U$ is in reduced echelon form, we call $U$ the reduced echelon form of $A$.

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- A pivot position in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A pivot column is a column of $A$ that contains a pivot position.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1/3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Example: Row reduce the matrix $A$ below to echelon form, and locate the pivot positions and pivot columns of $A$.

$$\begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

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- The pivot columns are C1, C2, C4, and the pivot positions are positions (1,1), (2, 2), and (3, 4).

# Row Redution Algorithm

- Example: Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

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- STEP 3: Use row replacement operations to create zeros in all positions below the pivot.

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# Row Redution Algorithm

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- STEP 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The combination of steps 1–4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

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# Solutions to Linear Systems

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- Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

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- There are 3 variables because the augmented matrix has four columns.
- The variables $x_1$ and $x_2$ corresponding to pivot columns in the matrix are called basic variables. The other variable, $x_3$, is called a free variable.

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- In the above example, solve the first and second equations for $x_1$ and $x_2$, we have $x_1 = 1 + 5x_3, x_2 = 4 - x_3$ and $x_3$ is free.
- Each different choice of $x_3$ determines a (different) solution of the system, and every solution of the system is determined by a choice of $x_3$.

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- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

  For example, in the above problem, we may take $x_2$ as free variables, and write $x_1$ and $x_3$ in terms of $x_2$:

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- When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

# Existence and Uniqueness Theorem

- Theorem (Existence and Uniqueness Theorem) A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form $[0 \ldots 0 b]$ with $b$ nonzero.

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- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

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3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

# Using Row Reduction to Solve a Linear System—Example

Ex: find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$