Section 1.3 Vector Equations

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- Two operations on vectors:
  - the sum of two vectors u and v is the vector $u + v$ obtained by adding corresponding entries of u and v.
  - given a vector u and a real number c, the scalar multiple of u by c is the vector cu obtained by multiplying each entry in u by c.

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$$4u + (-3)v = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}.$$

Geometric descriptions of $\mathbf{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

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- So we may regard $\mathbf{R}^2$ as the set of all points in the plane.
- If vectors u and v are represented as points in the plane, then the vector $u + v$ corresponds to the fourth vertex of the parallelogram whose other vertices are u, v and 0.

Vectors in $\mathbf{R}^3$ and $\mathbf{R}^n$

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- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbf{R}^n$ (read r-n) denotes the collection of all lists (or ordered n-tuples) of n real numbers, usually written as $n \times 1$ matrices, such as $\begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$

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  8. $1u = u$

Linear combinations of vectors

- Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$ and scalars $c_1, c_2, \ldots, c_p$, the vector y defined by

$$y = c_1 v_1 + c_2 v_2 + \ldots + c_p v_p$$

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- The weights in a linear combination can be any real numbers, including zero.

Linear combinations of vectors—Example

Ex: Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether vector b can be written as a linear combination of vectors a_1 and a_2.

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- This is a vector equations.

Solving vector equations—an example

- We can first rewrite the vector equation as a linear systems, by definitions of scalar multiplication and vector addition

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix},$$

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- So we have the following linear system

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \quad \text{with augmented matrix} \quad \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

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- Solve it, we get $x_1 = 3$ and $x_2 = 2$. So $b = 3a_1 + 2a_2$.

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- In particular, the vector b can be generated by a linear combination of vectors $a_1, a_2, \ldots, a_n$ if and only if there exists a solution to the linear system corresponding to the above augmented matrix.

Span of vectors

- Definition: If $v_1, \ldots, v_p$ are vectors in $\mathbf{R}^n$, then the set of all linear combinations of $v_1, \ldots, v_p$ is denoted by $Span\{v_1, \ldots, v_p\}$ and is called the subset of $\mathbf{R}^n$ spanned (or generated) by $v_1, \ldots, v_p$.

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- That is, $Span\{v_1, \ldots, v_p\}$ is the collection of all vectors that can be written in the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p,$$

with real numbers (or scalars) $c_1, c_2, \ldots, c_p$.

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- In general, let v be a nonzero vector in $\mathbf{R}^3$. Then $Span\{v\}$ is the set of all scalar multiples of v, which is the set of points on the line in $\mathbf{R}^3$ through vectors v and 0.
- Let u and v be vectors in $\mathbf{R}^3$. What is $Span\{u, v\}$?

1.4 Matrix equations (part 1)

Matrix equation

- Definition: If A is an $m \times n$ matrix, with column vectors $a_1, a_2, \ldots, a_n$, and if x is a vector in $\mathbf{R}^n$, then the product of A and x, denoted by Ax, is the linear combination of the column vectors of A using the corresponding entries in vector x as weights. That is

$$Ax = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n.$$

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- Ax is defined only if the number of columns of A equals the number of entries in vector x.

Examples

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- That is,

$$3v_1 - 5v_2 + 7v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = Ax.$$

Examples

- Ex 2: Write the following linear system as a vector equation involving a linear combination of vectors, and then as a matrix equation:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

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- As in the previous example, we may write it as matrix equation

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Equivalent formulations

- Theorem 3: if A is an $m \times n$ matrix, with column vectors $a_1, a_2, \ldots, a_n$, and if b is a vector in $\mathbf{R}^m$, then the following three equations have the same solution set

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  3. the linear system with augmented matrix
$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}$$
- So the equation $Ax = b$ has a solution if and only if vector b is a linear combination of the column vectors of A.

Existence of solutions—When vectors span $\mathbf{R}^m$?

- Theorem 4: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true or they are all false.

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  (b) Each b in $\mathbf{R}^m$ is a linear combination of the column vectors of A.

  (c) The column vectors of A span $\mathbf{R}^m$.

  (d) The matrix A has a pivot position in every row.

Example

- Ex: Determine if b is in the $Span\{v_1, v_2, v_3\}$, where vectors v_1, v_2, v_3, b are

$$v_1 = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Proof of Theorem 4

- As stated in the previous theorem, (a), (b) and (c) are all true or all false. So we just need to show (a) and (d) are either both true or false.

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- Let d be any vector with a 1 in its last entry. Then $\begin{bmatrix} U & d \end{bmatrix}$ is an inconsistent system.
- Since the row operations are reversible, $\begin{bmatrix} U & d \end{bmatrix}$ can be transformed into the form $\begin{bmatrix} A & b \end{bmatrix}$ for some vector b in $\mathbf{R}^m$.
- It follows that the system $Ax = b$ is inconsistent for that vector b. So (a) is false.