Section 1.7 Linear independence

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$$x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = 0 \tag{1}$$

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- A set of vectors is linearly dependent if and only if it is not linearly independent.

Example

- Ex 1: Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,

  a) determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

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- Solution: We must determine if there is a nontrivial solution to the following equation:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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- Row operations on the associated augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Choose any nonzero value for x_3, say $x_3 = 1$, we get $x_1 = 2, x_2 = -1, x_3 = 1$.
- So we obtain one (out of infinitely many) possible linear dependence relations among v_1, v_2, v_3:

$$2v_1 - v_2 + v_3 = 0$$

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- Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $Ax = 0$.
- Thus, the columns of matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

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- A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.

- Theorem 7: (Characterization of Linearly Dependent Sets) A set $S = \{v_1, v_2, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

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- So w is in $Span\{u, v\}$.

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- If $p > n$, there are more variables than equations, so there must be a free variable.
- Hence $Ax = 0$ has a nontrivial solution, and the columns of A are linearly dependent.
- Theorem 8 says nothing about the case in which the number of vectors in the set does not exceed the number of entries in each vector.

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- Then the equation $1v_1 + 0v_2 + \ldots + 0v_p = 0$ shows that S in linearly dependent.