Section 2.2 and 2.3 The Inverse of a Matrix

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Inverse Matrices

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- So we may denote the unique inverse of A by A^{-1}. So

$$A^{-1}A = AA^{-1} = I_n$$

Inverse of 2×2 matrices

- Theorem 4: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc = 0$, then A is not invertible, and if $ad - bc \neq 0$, then A is invertible and

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Soln. $A^{-1} = \frac{1}{3 \cdot 6 - 4 \cdot 5} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$.

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Soln. The solution is $x = A^{-1}b = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

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  (c) If A is invertible, then so is A^T, and
  $$(A^T)^{-1} = (A^{-1})^T.$$

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- If A in invertible, then A can be row reduced to an identity matrix.
- We now find A^{-1} by watching the row reduction of A to I.
- An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Ex. Compute E_1A, E_2A, E_3A, and describe how these product can be obtained by elementary row operations on A, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad A = \begin{bmatrix} a & b & v \\ d & e & f \\ g & h & i \end{bmatrix}$$

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- E_1A is the same as R3+(-4)R1, E_2A is the same as interchange R1 and R2, and E_3A is the same as 5R3.

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$$\text{if } E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ then } E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Then there are elementary matrices $E_1, E_2, \ldots, E_p$ so that $E_p \ldots E_2 E_1 A = I_n$.
- As E_is are invertible, their product is also invertible, so we have

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- Then $A^{-1} = E_p \ldots E_1$, which says that A^{-1} results from applying $E_1, \ldots, E_p$ successively to I_n.

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- So we have $A = (E_p \dots E_1)^{-1}$.
- It follows that $A^{-1} = ((E_p \dots E_1)^{-1})^{-1} = E_p \dots E_1$.
- Then $A^{-1} = E_p \dots E_1$, which says that A^{-1} results from applying $E_1, \dots, E_p$ successively to I_n.
- This is the same sequence that reduced A to I_n.

Algorithm to find A^{-1}

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- That is,

$$\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

Algorithm to find A^{-1}—-Example

Ex. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

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Soln. $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$.

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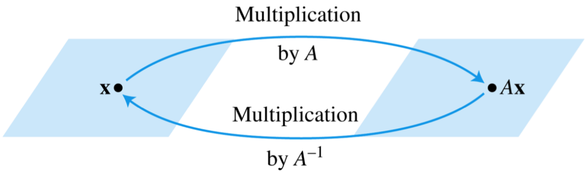
- See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

- A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be invertible if there exists a function $S : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$S(T(x)) = x \text{ for all } x \text{ in } \mathbf{R}^n \quad (1)$$

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- Theorem 9: Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(x) = A^{-1}x$ is the unique function satisfying equation (1) and (2).

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- Thus A is invertible, by the Invertible Matrix Theorem, statement (i).

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- Conversely, suppose that A is invertible, and let $S(x) = A^{-1}x$. Then, S is a linear transformation, and S satisfies (1) and (2).

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  (k) There is an $n \times n$ matrix D such that $AD = I$.

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  - (h) The columns of A span $\mathbf{R}^n$.
  - (i) The linear transformation $x \to Ax$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.
  - (j) There is an $n \times n$ matrix C such that $CA = I$.
  - (k) There is an $n \times n$ matrix D such that $AD = I$.
  - (l) A^T is an invertible matrix.

$a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d \Leftrightarrow e$

f

$g \Leftrightarrow h \Leftrightarrow i$

$a \Leftrightarrow l$

$a \Rightarrow j \Rightarrow d$:

$Ax = 0$ & let $CA = I$

$CAx = C \cdot 0 = 0 \Rightarrow Ix = 0 \Rightarrow x = 0$

$a \Rightarrow k \Rightarrow g$:

Let $AD = I$. Then $ADb = Ib = b$

so Db is a solution to $Ax = b$.

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- The Invertible Matrix Theorem applies only to square matrices.