Section 2.4–2.5 Partitioned Matrices and LU Factorization

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partition matrices into blocks

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- Ex: $A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right].$

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- In the block form, we have blocks $A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}$ and so on.

Operations on partitioned matrices

- (Addition and scalar multiplication) IF matrices A and B are the same size and are partitioned in exactly the same way, namely $A = (A_{ij})$ and $B = (B_{ij})$, then

$$A + B = (A_{ij} + B_{ij}), \text{ and } rA = (rA_{ij})$$

$$A+B = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] + \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}+B_{11} & \\ \hline & \end{array}\right]$$

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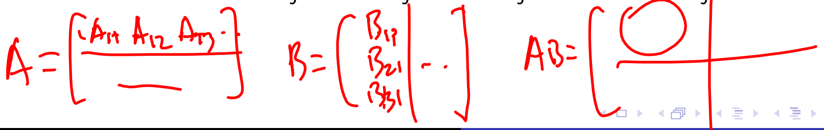
- (multiplication) If the partitions of A and B are comfortable for block multiplication, namely, the column partition of A matches the row partition of B, then if $A = (A_{ij})_{m \times n}$ and $B = (B_{ij})_{n \times p}$, then $AB = (C_{ij})_{m \times p}$, where

$$C_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj}$$



Example

- Find AB, where

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

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Soln:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$
$$= \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

Example

- Compute $\begin{bmatrix} A & O \\ I & B \end{bmatrix} \begin{bmatrix} C & I \\ O & D \end{bmatrix}$, where A, B, C and D are $n \times n$, O is the $n \times n$ zero matrix, and I is the $n \times n$ identity matrix.

$$\begin{bmatrix} A & O \\ I & B \end{bmatrix} \begin{bmatrix} C & I \\ O & D \end{bmatrix} = \begin{bmatrix} AC & A \\ C & I+BD \end{bmatrix}$$

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- If we do direct multiplication: to compute each term uses $2n$ multiplications and $2n$ additions, or $4n$ flops (floating point operations). Since there are $4n^2$ terms in the resulting matrix, direct multiplication uses $4n \cdot 4n^2 = 16n^3$ flops.

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- In the above computation, to compute AC takes $2n \cdot n^2$ flops, to compute BD takes $2n \cdot n^2$ flops, to add $I + BD$ takes n flops, so partitioned multiplication uses $4n^3 + n$? flops.

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- Take $n = 1000$. How many steps do we save?

Inverse of Partitioned Matrices

- A matrix of the form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is said to be block upper triangular. Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

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Soln: Let A^{-1} be B and partition B so that

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- So we will have

$$A_{11}B_{11} + A_{12}B_{21} = I_p \quad (1)$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \quad (2)$$

$$A_{22}B_{21} = 0 \quad (3)$$

$$A_{22}B_{22} = I_q \quad (4)$$

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- From (2), we get $A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1}$. So $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$.

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$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

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- A block diagonal matrix is a partitioned matrix with zero blocks off the main diagonal (of blocks). From the above example, such a matrix is invertible if and only if each block on the diagonal is invertible.

Example: find the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{4}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \frac{1}{3} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \frac{1}{8}\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} \end{bmatrix}$$

LU Factorization

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- In particular, if A can be row reduced to echelon form without row interchanges, then we could make the main diagonal of L to be 1s. Or

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ l_{m1} & l_{m2} & l_{m3} & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \dots & & & & \\ 0 & 0 & 0 & \dots 0 & u_{nm} \end{bmatrix}$$

L m×m echelon form

Motivation for LU Factorization

If we want to solve the matrix equation $Ax = b$ and we have LU-factorization of $A = LU$, then we can the following:

- $Ax = LUx = L(Ux) = b$ so let $y = Ux$, we have $Ly = b$.
- Solve $Ly = b$ to get y.
- Sove $Ux = y$ to get x.

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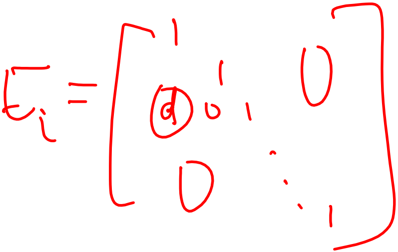

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- Then $A = (E_p \ldots E_1)^{-1} U = LU$, with

$$L = (E_p \ldots E_1)^{-1} = E_1^{-1} \ldots E_p^{-1} I$$

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  2. Then $L = E_1^{-1} \ldots E_p^{-1}$. Or $I = E_1 \ldots E_p L$.

Example

Ex. Find an LU factorization of $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

$$\xrightarrow[\substack{R3-R1 \\ R4+3R1}]{R2+2R1} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \xrightarrow[R4-4R2]{R3+3R2} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

$$\xrightarrow{R4-2R3} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} = L$$

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- We first reduce A to an upper triangular form by row operations, and record each step with an elementary matrix.

$$A \to \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1 \tag{5}$$

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- In the first step, we used row operations $R2 + 2R1$, $R3 - R1$, $R4 + 3R1$, which corresponds to elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}.$$

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- In the second step, we used row operations $R3 + 3R2$, $R4 - 4R2$, which corresponds to elementary matrices

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

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- In the last step, the row operation is $R4 - 2R3$, and the corresponding elementary matrix is $E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$,

- One important observation (the position of d could be anywhere):

$$\text{if } E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ then } E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- So (pay attend to the result)

$$L_1 = E_1^{-1} E_2^{-1} E_3^{-1} I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

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- Thus

$$L = L_1E_4^{-1}E_5^{-1}E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$