Section 3.1 Introduction to Determinants

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Definition of Determinant

- Recall that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its determinant $\det(A) = ad - bc$ is non-zero.

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- Let's take a look at the 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \to \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

  where $ad - bc$ is the determinant of the matrix.

- So 3×3 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

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- If A is invertible, then Δ must be nonzero. We will see that the converse is also true.
- We call the Δ to be the determinant of the 3×3 matrix A.

- We could group the terms in Δ and get that

$$\Delta = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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- Definition: For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) + \ldots + (-1)^{n+1} a_{1n} \det($$

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- instead of $\det(A)$, sometimes we also use $|A|$ to denote the determinant of A.

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- Ex2: compute the determinant of $A_2 = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

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- Soln: it is complicated....
- BUT it should not be, as if you look at the first column instead of first row....

Another definition

- Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

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- Theorem 1: the determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or any column. So

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

and

$$\det(A) = a_{1j}C_{ij} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

- So in the previous example

$$\det(A_2) = 3\det\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 3 \cdot 2 \cdot \det\begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & -1 \\ 0 - 2 & 0 & \end{bmatrix}$$

$$= 3 \cdot 2 \cdot 1 \cdot \det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} = 3 \cdot 2 \cdot 1 \cdot (4 \cdot 0 - (-1) \cdot (-2)) = -12$$

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- Theorem 2: if A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

$$\det\begin{bmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} = a_1 a_2 \cdots a_n, \qquad \det\begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ * & & & a_n \end{bmatrix} = a_1 a_2 \cdots a_n$$

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- Theorem 2: if A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

- When we use row operations on a matrix, how does the determinant change? Note that we can always reduce it to a triangular matrix, which is easy to find its determinant.

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$\det(A) = \frac{1}{k} \cdot \det(B)$

- This theorem provides an efficient way to compute the determinant of a matrix.

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- Soltion: $|A| = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$

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Ex4: Compute det(A), where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

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- Solution:

$$|A| = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1)$$

Invertible matrix and determinant

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- So we have the following

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- Thoerem 4: A square matrix is invertible if and only if $\det(A) \neq 0$.

Column operations

- Theorem 5: If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

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Ex: For $n \times n$ matrices A and B, show that A is singular if $\det(B) \neq 0$ and $\det(AB) = 0$.

i.e. A is not invertible.

Soln: As $\det(AB) = (\det(A))(\det(B)) = 0$, $\det(A) = 0$ or $\det(B) = 0$.

- Since $\det(B) \neq 0$, $\det(A) = 0$. That is, A is singular.