# Section 3.3 Cramer's Rule, Volume, and Linear Transformations

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- Theorem (Cramer's Rule) Suppose that $A$ is an $n \times n$ invertible matrix. For any $b \in \mathbf{R}^n$, the unique solution to $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

# Examples

Ex: Use Cramer's rule to solve $Ax = b$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and

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- So $\det(A) = 11, \det(A_1(b)) = 55, \det(A_2(b)) = 77$.
- Therefore $x_1 = \frac{55}{11} = 5$ and $x_2 = \frac{77}{11} = 7$, and $x = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

# Proof of Cramer's Rule

- We have

$$\begin{aligned} A \cdot I_i(x) &= \begin{bmatrix} Ae_1 & Ae_2 & \dots Ae_{i-1} & Ax & Ae_{i+1} & \dots & Ae_n \end{bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & \dots a_{i-1} & b & a_{i+1} & \dots & a_n \end{bmatrix} \\ &= A_i(b) \end{aligned}$$

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- The theorem follows from that fact that $\det(I_i(x)) = x_i$.

# An inverse formula

- Suppose that $A$ is an $n \times n$ matrix. We define the $n \times n$ adjoint of $A$ as

$$Adj(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & & & \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

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Ex. Compute $A^{-1}$ if $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

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- So the area is $\left| \begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} \right| = |-28| = 28$.

# Area of triangles

- Theorem: suppose that points $P = (x_1, y_1), Q = (x_2, y_2)$ and $R = (x_3, y_3)$ form a triangle. The area of the triangle $PQR$ is

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- The area of the triangle is half of the area of the parallelogram.

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- To see this, if $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then $x = Au = \begin{bmatrix} x_1/a \\ x_2/b \end{bmatrix}$. So $u_1^2 + u_2^2 = 1$, which means $u$ is on the unit circle.

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- Therefore, $Area(E) = |\det(A)| \cdot Area(D) = ab \cdot \pi(1)^2 = \pi ab$.