Section 4.1 Vector Spaces

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  10. $1u = u$. (one)

$(-1)u = -u$

$1 \cdot u = u$

Examples

Ex1 $\mathbf{R}^n$ is a vector space. (check the ten rules)

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \begin{matrix} u+v \\ c\cdot u \end{matrix}$$

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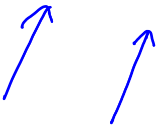
Ex2 Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.



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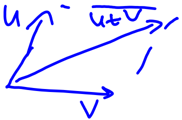

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Ex3 For $n \geq 0$, the set P_n of polynomials of degree at most n. So P_n consists of all polynomials of the form $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$, where coefficients $a_0, a_1, \ldots, a_n$ and the variable t are real numbers.

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$$(p+q)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n$$

and scalar multiplication as

$$(cp)(t) = cp(t) = ca_0 + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n$$

$p(t) = q(t)$: $\Longleftrightarrow$ $a_i = b_i$ for all i

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This is a vector space (of polynomials of degree at most n)

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- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).
- The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{0\}$.

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$\begin{bmatrix}1\\2\end{bmatrix} \notin R^3$

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Ex. The set $H = \{(s, t, 0)^T : s, t \in \mathbf{R}\}$ is a subset of $\mathbf{R}^3$. And it is a subspace of $\mathbf{R}^3$.

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Ex. A plane in $\mathbf{R}^3$ not through the origin is not a subspace of $\mathbf{R}^3$.

A subspace spanned by a set

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Proof. we need to verify the three (four) conditions:

$H \subseteq V$ H is a subset of V.

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Proof. we need to verify the ~~three~~ four conditions: $H \subseteq V$.

- Zero is in H, as $0 = 0v_1 + 0v_2$.
- If $u, w \in H$, then $u = s_1v_1 + s_2v_2$ and $w = t_1v_1 + t_2v_2$ for some $s_1, s_2, t_1, t_2 \in \mathbf{R}$.
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- For any $c \in \mathbf{R}$ and $u \in H$, we have $cu = (cs_1)v_1 + (cs_2)v_2 \in H$.

- So H is a subspace of V.

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- Given any subspace H of V, a spanning (or generating) set for H is a set $\{v_1, \ldots, v_p\}$ in H such that $H = Span\{v_1, \ldots, v_p\}$.

Example

Ex. Let $H = \{(a - 3b, b - a, a, b)^T : a, b \in \mathbf{R}\}$. That is, H is the set of all vectors of the form $(a - 3b, b - a, a, b)^T$ where a and b are arbitrary scalars. Show that H is subspace of $\mathbf{R}^4$.

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Proof. The vectors in H can be written as linear combinations:

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

v_1 v_2

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- So $H = Span\{v_1, v_2\}$ with $v_1 = (1, -1, 1, 0)^T$ and $v_2 = (-3, 1, 0, 1)^T$. Thus H is a subspace of $\mathbf{R}^4$.

Example

- For what values of h will y be in the subspace of $\mathbf{R}^3$ spanned by v_1, v_2, v_3 if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

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Sol. let $y = x_1 v_1 + x_2 v_2 + x_3 v_3$ with $x_1, x_2, x_3 \in \mathbf{R}$. We then have a linear system whose argumented matrix

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

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- By looking at its echelon form, we see that the linear system is consistent only if $h - 5 = 0$.

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$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Sol. let $y = x_1 v_1 + x_2 v_2 + x_3 v_3$ with $x_1, x_2, x_3 \in \mathbf{R}$. We then have a linear system whose argumented matrix

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

- By looking at its echelon form, we see that the linear system is consistent only if $h - 5 = 0$.
- So y is in H if $h = 5$.