Section 5.3 Diagonization

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Diagonalizable Matrices

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- Is A similar to D?
- Sometimes!
- A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

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$$\det(P) = 1\cdot(-2) - 1\cdot(-1) = -1 \neq 0$$

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = A$$

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- Find a formula for A^k.

$$A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$

$$= P\cdot D\cdot D\cdots D\cdot P^{-1} = PD^kP^{-1}$$

$$D^k = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\cdots\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$$

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- But $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$.
- So

$$A^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

The Diagonalization Theorem

- Theorem 5: An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

$$A = PDP^{-1} \iff AP = PD$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

$$\iff A[v_1\ v_2 \cdots v_n] = [v_1 \cdots v_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$P = [v_1\ v_2 \cdots v_n]$$

$$\| \quad [Av_1\ Av_2 \cdots Av_n] = [\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n]$$

$$Av_i = \lambda_i v_i \quad \forall i \iff$$

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- In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbf{R}^n$. We call such a basis an eigenvector basis of $\mathbf{R}^n$.

Proof of the Diagonalization Theorem

PF. First, if P is any $n \times n$ matrix with columns $v_1, \ldots, v_n$, and D is any diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$AP = A[v_1 \;\; v_2 \;\; \ldots \;\; v_n] = [Av_1 \;\; Av_2 \;\; \ldots \;\; Av_n]$$

and

$$PD = P \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} = [\lambda_1 v_1 \;\; \lambda_2 v_2 \;\; \ldots \;\; \lambda_n v_n]$$

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- Since P is invertible, the columns $v_1, v_2, \ldots, v_n$ of P must be linearly independent and non-zero.

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- It follows that $Av_i = \lambda_i v_i$ for each i.
- Since P is invertible, the columns $v_1, v_2, \dots, v_n$ of P must be linearly independent and non-zero.
- So λ_i are eigenvalues and v_i are the corresponding eigenvectors.

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- Let $P = [v_1 \ v_2 \ \ldots \ v_n]$ and D be the diagonal matrix with entries λ_i's.
- Then we have $AP = PD$.
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- Then we have $AP = PD$.
- Furthermore, as the columns of P are linearly independent, P is invertible.
- So $AP = PD$ implies that $A = PDP^{-1}$.

Example

Ex. Diagonalize the following matrix, if possible. (In other words, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda)(1-\lambda) + (-3)\cdot 3\cdot 3 + 3\cdot(-3)\cdot 3 - 3\cdot(-5-\lambda)\cdot 3 - (1-\lambda)\cdot 3\cdot(-3) - 3\cdot(-3)(1-\lambda)$$

$$= (1-\lambda)^2(-5-\lambda) - 9\left[+3 + 3 + (-5-\lambda) - (1-\lambda) - (1-\lambda)\right]$$

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- basis for $\lambda = 1$: $v_1 = [1 \ -1 \ 1]^T$.

$\lambda = 1$: $(A - 1\cdot I)x = 0$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R3+R2} \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 3 & 0 & -3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Now we find the three linearly independent eigenvectors of A:
- basis for $\lambda = 1$: $v_1 = [1 \; -1 \; 1]^T$.
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$$(A-(-2)I)\,v = 0$$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_1 = -x_2 - x_3$$

$$x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$$

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- Then we construct P from the vectors in the above step:

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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- Note that it is a good idea to check that $AP = PD$ (not required).

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- So A is not diagonalizable.

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So A is diagonalizable.

- The above theorem provides a sufficient condition for a matrix to be diagonalizable.
- However, it is not necessary for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.

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- The basis for the eigenvalues are
  basis for $\lambda = 5$: $v_1 = [-8\ 4\ 1\ 0]^T$ and $v_2 = [-16\ 4\ 0\ 1]^T$
  basis for $\lambda = -3$: $v_3 = [0\ 0\ 1\ 0]^T$ and $v_4 = [0\ 0\ 0\ 1]^T$.

$$A-(-3)I = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 0$
x_3
x_4

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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- We can check that the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

$$\det\begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = 1\cdot\det\begin{bmatrix} -8 & -16 & 0 \\ 4 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= 1\cdot 1\cdot\det\begin{bmatrix} -8 & -16 \\ 4 & 4 \end{bmatrix} \neq 0$$

Example

Ex. Diagonalize $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.

- As A is a triangular matrix, the eigenvalues are 5 (with multiplicity 2) and -3 (with multiplicity 2).
- The basis for the eigenvalues are
  basis for $\lambda = 5$: $v_1 = [-8\ 4\ 1\ 0]^T$ and $v_2 = [-16\ 4\ 0\ 1]^T$
  basis for $\lambda = -3$: $v_3 = [0\ 0\ 1\ 0]^T$ and $v_4 = [0\ 0\ 0\ 1]^T$.
- We can check that the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.
- So the matrix $P = [v_1\ v_2\ v_3\ v_4]$ is invertible and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Matrices whose eigenvalues are not distinct

THM. Let A be an $n \times n$ matrix whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

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|| #vectors in a basis.

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- The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n.

vectors in the bases.

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- For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is at most the the multiplicity of the eigenvalue λ_k.
- The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n.
- the sum of the dimensions of the eigenspaces equals n if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

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- For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is at most the the multiplicity of the eigenvalue λ_k.
- The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n.
- the sum of the dimensions of the eigenspaces equals n if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.
- If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, B_2, \ldots, B_p$ forms an eigenvector basis for $\mathbf{R}^n$. # is n.