# Section 6.4-6.5 The Gram-Schmit Process, least-square problems, and applications to linear models

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# Example

Ex. Let $x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $x_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, and $x_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $y = \begin{bmatrix}2\\-1\\3\\4\end{bmatrix}$, and $W = Span\{x_1, x_2, x_3\}$. Find the projection of $y$ onto $W$.

$$\text{Proj}_W \hat{y} = \frac{y \cdot x_1}{x_1 \cdot x_1} x_1 + \frac{y \cdot x_2}{x_2 \cdot x_2} x_2 + \frac{y \cdot x_3}{x_3 \cdot x_3} x_3$$

WRONG! not orthogonal set!

# Orthogonal basis

- Given a basis for a subspace $W$ of $\mathbf{R}^n$, how to find an orthogonal basis for $W$?
- Let $\{x_1, x_2, \ldots, x_p\}$ be a basis for $W$.
- The idea is as follows: let $v_1 = x_1$ and take $W_1 = Span\{v_1\}$, then project $x_2$ to $W_1$ and let $v_2$ be the component of $x_2$ orthogonal to $W_1$; then let $W_2 = Span\{v_1, v_2\}$, and project $x_3$ to $W_2$ and let $v_3$ be the component of $x_3$ orthogonal to $W_2$. Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for $W$; and so on.
- This is so-called Gram-Schmidt Process.

# The Gram-Schmidt Process

- Given a basis $\{x_1, x_2, \ldots, x_p\}$ for a nonzero subspace $W$ of $\mathbf{R}^n$, define

$$
\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \quad = x_2 - \mathrm{Proj}_{W_1} x_2 \\
v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \quad = x_3 - \mathrm{Proj}_{W_2} x_3 \\
&\cdots \\
v_p &= x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \ldots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}
\end{aligned}
$$

Then $\{v_1, v_2, \ldots, v_p\}$ is an orthogonal basis for $W$.

# Example

Ex. Let $W = span\{x_1, x_2\}$, where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{v_1, v_2\}$ for $W$.

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$$v_2 = x_2 - p = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

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- $v_3 = x_3 - proj_{W_2} x_3 = x_3 - (\frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2) = \begin{bmatrix} -3/8 \\ -13/24 \\ 11/24 \\ 11/24 \end{bmatrix}$.

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- Definition: If $A$ is $m \times n$ and $b$ in $\mathbf{R}^n$, a least-squares solution of $Ax = b$ is an $\hat{x}$ in $\mathbf{R}^n$ such that $||b - A\hat{x}|| \leq ||b - Ax||$ for all $x \in \mathbf{R}^n$.

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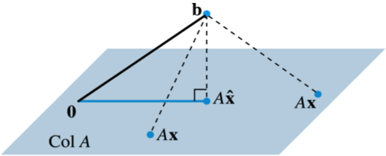


The vector **b** is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other **x**.

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

$\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$

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least-square solution to $Ax = b$

$\Updownarrow$

find $\hat{x}$ s.t. $A\hat{x} = Proj_{Col A} b$

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- Such an $\hat{x}$ in $\mathbf{R}^n$ is a list of weights that will build $\hat{b}$ out of the columns of $A$.
- Suppose $\hat{x}$ satisfies $A\hat{x} = \hat{b}$.
- By the Orthogonal Decomposition Theorem, the projection $\hat{b}$ has the property that $b - \hat{b}$ is orthogonal to $Col\ A$, so $b - A\hat{x}$ is orthogonal to each column of $A$.

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- A solution to $A^TAx = A^Tb$ is often denoted by $\hat{x}$.

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- Since the columns of $A$ span *Col* $A$, the vector $b - A\hat{x}$ is orthogonal to all of *Col* $A$.
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- By the uniqueness of the orthogonal decomposition, $A\hat{x}$ must be the orthogonal projection of $b$ onto $Col\ A$.
- That is, $A\hat{x} = \hat{b}$ and $\hat{x}$ is a least-squares solution.

# Example

Ex. Find a least-squares solution of the inconsistent system $Ax = b$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

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$A^T A x = A^T b$

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- Solve it, we have $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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  When these statements are true, the least-squares solution $\hat{x}$ is given by $\hat{x} = (A^T A)^{-1} A^T b$.

- The distance from $b$ to $A\hat{x}$, $||b - A\hat{x}||$, is called the least-squares error of this approximation.

# Alternative calculation

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- When the columns of $A[u_1 \ u_2 \ \ldots \ u_p]$ are orthogonal, we know exactly the orthogonal projection of $b$ on $Col\ A$:

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- Such matrices often appear in linear regression problems.

# Example

Ex. Find a least-squares solution of $Ax = b$ for

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

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- We can also get the least-squares error: $||b - A\hat{x}|| = \sqrt{\ }$.

Applications to linear models

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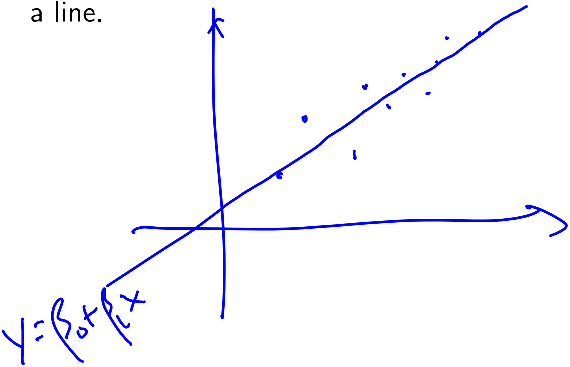
- The simplest relation between two variables $x$ and $y$ is the linear equation $y = \beta_0 + \beta_1 x$. Experimental data often produce points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ that, when graphed, seem to lie close to a line.



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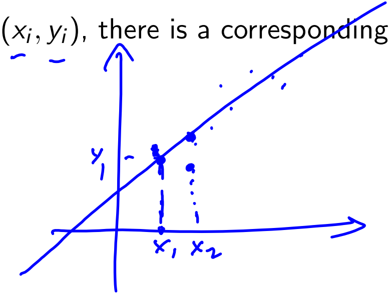


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- We call $y_i$ the observed value of $y$ and $\beta_0 + \beta_1 x_i$ the predicted $y$-value. The difference of an observed $y$-value and the predicted $y$-value is called a residual.
- The least-squares line, or the line of regression of $y$ on $x$, is the line $y = \beta_0 + \beta_1 x$ that minimizes the sum of the squares of the residuals.

- The $\beta_0$ and $\beta_1$ satisfy the following

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \dots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

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- The least-squares line is $y = 2/7 + 5/14x$.