Math 432 Homework Twelve

Due: Friday, April 22, 2016

Prove the following statements. Four points for each.

(1) For $n \in N$ and $1 \leq k \leq n-1$, define a square $A^{(k)}$ by $a^k_{i,j} = ki + j \mod n$.
(a) Prove that $A^{(k)}$ is a Latin square if and only if n and k are relatively prime.
(b) When $A^{(k)}$ and $A^{(l)}$ are Latin squares, prove that they are orthogonal if and only if $k - l$ is relatively prime to n.

(2) Le M be a Latin square that can be written as $\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ with X and Y being Latin squares of odd order. Prove that M has no transversal, where a transversal is a set of n distinct entries in distinct rows and distinct columns. Use this to prove that there is no Latin square orthogonal to M.

(3) (i) Determine the index of the set $\{2, 4, 6, 7\}$ in the colex ordering. (ii) Find the 4-binomial expansion of the integer 40.

(4) Suppose that $m = \binom{r}{k}$. Let F be a family with minimum shadow among the k-uniform families of size m in the subsets of $[n]$. Prove that F consists of all k subsets among some r elements. (Hint: apply the Kruskal-Katona Theorem)

(5) Let p, r, s, t be integers with $2 \leq p < r$. Suppose also that $r \leq t+1 \leq s$ or that $t = 0$ and $r \leq s$. Use Kruskal-Katona Theorem to prove that every graph with at most $\binom{s}{p} + \binom{t}{p-1}$ cliques of size p has at most $\binom{s}{r} + \binom{t}{r-1}$ cliques of size r. Show that the bound is sharp.