Math 432 – Combinatorics
Homework 4
Due Feb 19, 2016.

Work the following problems. Show all your work. Each problem is 4 points.

1. Use generating function to evaluate the following sum:

$$\sum_{k=1}^{n} k \binom{n}{k}^2.$$

2. In how many ways can one pick 25 coins that are pennies, nickels, or dimes, with at least three nickels, at most five dimes and an even number of pennies?

3. (B2, Putnam 1992) For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

4. Prove the identity below. (hint: choose x appropriately in $\sum \binom{2n}{n} x^n = (1-4x)^{-1/2}$.)

$$\sum (-1)^k \binom{n-k}{k}\binom{2n-2k}{n-k} = 2^n$$

5. Evaluate the sum below using convolution, and give a combinatorial proof of the resulting identity.

$$\sum_{j=0}^{k} \binom{n+k-j-1}{k-j}\binom{m+j-1}{j}.$$

6. For positive integers m, n, using generating function to prove that

$$\sum_{k} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$