Math 432 Lec 03 Binomial Formula, Combinatorial Identities, and counting models

(1) Binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Prove it in two ways (combinatorial proof and induction proof)

(2) Identities involving binomial coefficients:

(a) $\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} = 2^{n-1}$.

Two proofs: set $x = 1, y = -1$ in Binomial Formula; or use the formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ to show that the sums are just the sums of numbers in the previous row in Pascal Triangle.

(b) $k\binom{n}{k} = n\binom{n-1}{k-1}$, and more general $\binom{k}{l}\binom{n}{k} = \binom{n}{l}\binom{n-l}{k-l}$.

The first one has two methods: count k-member committees with a chair in two ways, or use binomial formula (differentiate both sides). The first method can be used to prove the second identity.

(c) $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$, and more general, $\sum_k \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$. (Vandermonde Convolution)

A sum in an identity suggests that we count something in cases. So the first method is to count the n-subsets of $[2n]$ in cases. The key is to come out with a right criterion to break them into parts.
You may also try to use the Binomial Formula to prove the first one.

(d) $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$
We count the $r+1$-subsets of $[n+1]$ with the largest element $k+1$, $r \le k \le n$.

(3) **Classic models: words, sets, and multisets**

When counting an object, we first have to figure out two factors: order and repetition. If order of elements doesn't matter, then it is either a set (repetition not allowed) or a multi-set (repetition allowed). If order matters, then it is either a simple word (or simple tuple; repetition not allowed) or a word (repetition allowed).

(a) k**-sets:** there are $\binom{n}{k}$ k-sets in an n-set. Remark: $\binom{n}{k}$ is called **binomial coefficients**.

(b) k**-multisets:** Choose a k-element multiset (repetition allowed) from $[n]$.

Theorem: the number of k-multisets from $[n]$ equals to the number of nonnegative integer solutions to $\sum_{i=1}^{n} x_i = k$, which equals to $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$.

Example: What is the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 10$ in which $x_1 \geq 3, x_2 \geq 1, x_3 \geq 0$ and $x_4 \geq 5$?

(c) k**-word:** there are n^k k-words from an alphabet S of size n.

(d) **simple** k**-word:** there are $n_{(k)}$ simple k-words from an alphabet of size n.

Example: How many 4-words can one form from the letters in CLASSIC?

The answer is $5 \cdot 4 \cdot 3 \cdot 2 + \binom{4}{2} \cdot 4 \cdot 3 \cdot 2 + \binom{4}{2}$. (why?)