Math 432 Lec 07 Generating function -1

(1) Operation on generating functions.

Let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ be two generating functions.
Addition: $A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n$ (count by cases)

Multiplication: $A(x)B(x) = \sum_{n\geq 0}(\sum_{k=0}^n a_{n-k}b_k)x^n$ (count by steps). This is also called *convolution* of two generating functions.

Let $C(x) = \sum_{n\geq 0} C_n x^n$. Then

$$C_n = \sum_{k=0}^{n} a_{n-k}b_k \text{ if and only if } C(x) = A(x)B(x)$$

(2) Solve selection problems with restrictions.

We know from Extended Binomial Formula that $(1-x)^{-k} = \sum_{n\geq 0}\binom{k+n-1}{k-1}x^n$. But we can also see this from a multiplication of n of $(1-x)^{-1}$. Suppose that we choose x^{a_i} from i-th $(1-x)^{-1}$, then the coefficient of x^n in $(1-x)^{-k}$ is the nonnegative integer solutions to $a_1 + a_2 \ldots + a_k = n$, which we know is $\binom{n+k-1}{k-1}$.

This idea helps to solve selection problems with more restrictions.

Ex: Find the number of nonnegative integer solutions to $x_1+x_2+x_3+x_4 = n$ so that $4|x_1, 3|x_2$ and x_2 is even.

By let $x_1 = 4a_1, x_2 = 3a_2, x_3 = 2a_3$ and $x_4 = a_4$, we ask for a_n , the number nonnegative integer solutions to $4a_1 + 3a_2 + 2a_3 + a_4 = n$. Note that a_n is the coefficient of the generating function

$$\begin{aligned}
& (1+x^4+x^8+\ldots)(1+x^3+x^6+\ldots)(1+x^2+x^4+\ldots)(1+x+x^2+\ldots) \\
= \; & (1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1} = (1-x)^{-4}(1+x)^{-2}(1+x^2)^{-1}(1+x+x^2)^{-1} \\
= \; & \frac{A_1}{1-x} + \frac{A_2}{(1-x)^2} + \frac{A_3}{(1-x)^3} + \frac{A_4}{(1-x)^4} + + \frac{B_1}{(1+x)} + \frac{B_2}{(1+x)^2} + \frac{C_1x + c_2}{1+x^2} + \frac{D_1x + D_2}{1+x+x^2}
\end{aligned}$$

(3) Solve recurrence relations

Ex. The Catalan number C_n satisfies the recurrence relations $C_n = \sum_{k=1}^n C_{k-1}C_{n-k}$ for $n \geq 1$, with $C_0 = 1$. Find the general formula for C_n only in terms of n.

From $C_n = \sum_{k=1}^n C_{k-1}C_{n-k}$, we multiply both sides with x^n and then sum over all $n \geq 1$, we get $\sum_{n\geq 1} C_n x^n = \sum_{n\geq 1}\sum_{k=1}^n C_{k-1}C_{n-k}x^n$. The LHS is

$C(x)-1$. For RHS, we first make a substitution $\ell = k-1$ and then a substitution $m = n-1$, we get

$$C(x)-1 = LHS = RHS = x \cdot \sum_{m\geq 0}\sum_{\ell=0}^{m} C_\ell C_{m-\ell} x^m = x(C(x))^2.$$

So $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. Note that

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n = 1 + \sum_{n\geq 1}\frac{-2}{n}\binom{2(n-1)}{n-1}x^n$$

So $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ (if take $+$, then there is $1/x$ in $C(x)$, which is not allowed), and

$$C(x) = \sum_{n\geq 1}\frac{1}{n}\binom{2(n-1)}{n-1}x^{n-1} = \sum_{m\geq 0}\frac{1}{m+1}\binom{2m}{m}x^m.$$

Take the coefficient of x^n, we get $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(4) Generating function for recurrences with multiple indices, for example, $a_{n,k}$.

We fix n, and get $A_n(x) = \sum_{k\geq 0} a_{n,k}x^k$. Then let $B(x,y) = \sum_{n\geq 0} A_n(x)y^n$.

Ex: Let $a_{n,k} = a_{n-1,k-1} + a_{n-1,k}$ for $n \geq 1$ and $a_{0,0} = 1, a_{0,k} = 1$ for $k \geq 1$. (Recurrence for Pascal's triangle)

We can first get $A_n(x) = (1+x)^n$, and then get $B(x,y) = (1-(1+x)y)^{-1}$.