Math 432 lec 09 Principle of Inclusion-Exclusion

Let $P_1, P_2, \ldots, P_m$ be m properties referring to the objects in universe S, and let

$$A_i = \{x : x \in S \text{ and } x \text{ has property } P_i\ \}, (i = 1, 2, \ldots, m)$$

be the subset of objects of S that have property P_i (and possibly other properties).

Theorem: The number of objects of S that have none of the properties $P_1, P_2, \ldots, P_m$ is given by

$$|\overline{A_i} \cap \overline{A_i} \cap \ldots \cap \overline{A_m}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| + \ldots + (-1)^m |A_1 \cap A_2 \cap \ldots \cap A_m|,$$

where the i^{th} sum is over all i-combinations of $\{1, 2 \ldots, m\}$.

Proof: The left side counts the number of objects of S with none of the properties. We can establish the validity of the equation by showing that an object with none of the properties makes a net contribution of 1 to the right side, and an object with at least one of the properties makes a net contribution of 0.

Assume that the size of the set $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ that occurs in the inclusion-exclusion principle depends only on k and not on which k sets are used in the intersection. Suppose that $\alpha_k = |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$. Then

$$|\cap_{i=1}^m \overline{A_i}| = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \alpha_k.$$

Examples:

(1) How many permutations of the letters

$$M, A, T, H, I, S, F, U, N$$

are there such that none of the words $MATH$, IS, and FUN occur as consecutive letters.

(2) (*Derangements*)
For $n \geq 1$, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!})$.

Remark: since $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} + \ldots$, we have $e^{-1} = \frac{D_n}{n!} + (-1)^{n+1} \frac{1}{(n+1)!} + \ldots$. Thus when n is not so small ($n \geq 7$), e^{-1} is very close to $\frac{D_n}{n!}$, that is, the probability to have a derangement is close to $\frac{1}{e}$ when $n \geq 7$.

(3) (*Permutations with forbidden positions*)

Let $X_1, X_2, \ldots, X_n$ be (possibly empty) subsets of $[n]$. Define $P(X_1, X_2, \ldots, X_n)$ to be the set of all permutations of $[n]$ such that the number in i^{th} position is not in X_i. Let $p(X_1, X_2, \ldots, X_n) = |P(X_1, X_2, \ldots, X_n)|$. Then

$$p(X_1, X_2, \ldots, X_n) = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$$

where r_k is the number of ways to place k non-attacking rooks on the n-by-n board such that each of the k rooks is in a forbidden position, $(k = 1, 2, \ldots, n)$. Note that X_i gives the forbidden positions $\{(i, j) : j \in X_i\}$.

Example: Determine $p(\{1\}, \{1, 2\}, \{3, 4\}, \{3, 4\}, \emptyset, \emptyset)$.

(4) (*Permutations with relative forbidden positions*) Let Q_n be the number of permutations containing no patterns $12, 23, 34, \ldots, (n-1)n$. Then

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$

(5) (*Proving identities*)

For a sum of the form $\sum_{k=0}^{n} (-1)^k \binom{n}{k} c_k$, it may count some objects using the inclusion-exclusion method. If it is the case, then c_k counts the number of objects with at least k properties. The terms $k = 0$ and $k = 1$ will suggest the universe and the sets within it.

Examples: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+n-k}{p-k} = \binom{m}{p}$.

Solution: count p-sets in $[m]$, which are the p-sets in $[m+n]$ that use none of the extra elements.