Math 431 Lecture 10: Exponential Generating Functions

The **exponential generating function (EGF)** for a series $a_0, a_1, \ldots, a_n, \ldots$ of complex numbers is the formal power series $g(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$. Note that if $a_n = 1$ for all n, then

$$g(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x.$$

- OGF is usually used for set/multiset problems (selection problems), while EGF is usually used for word/multiword problems (permutation problems), in which order is important.
- The product of EGFs corresponds to forming order/label structures in steps, where the steps are described by allocation of labels.

  Theorem: Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$, where $n_1, n_2, \ldots, n_k$ are non-negative integers or infinite. Let h_n be the number of n-words of S. Then the EGF for $h_0, h_1, \ldots, h_n, \ldots$ is

$$g(x) = f_{n_1}(x) f_{n_2}(x) \ldots f_{n_k}(x), \text{ where } f_{n_i}(x) = \sum_{k=0}^{n_i} \frac{x^k}{k!}.$$

Examples:

(1) *n-ary words of length k.* The labels are the k positions and we allocate k labels to n sets. The EGF is $\sum_{k=0}^{\infty} n^k \frac{x^k}{k!} = e^{nx}$. On the other hand, the EGF is the product of the EGFs associated with each letter, and each of those EGFs is e^x.

(2) *Words with restricted usage of letters.*
- When no restriction for multiplicity, the EGF for each letter is e^x.
- When each letter must be used, the EGF for each letter is $e^x - 1$.
- If each letter is used to at most once, the EGF is $1 + x$. Thus the EGFs for simples words from k letters is $(1 + x)^k$, and the number of simple words of length n formed from $[k]$ is $k(k-1)\ldots(k-n+1) = k_{(n)}$.
- If a letter is used even times, then the EGF is

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = 0.5(e^x + e^{-x}).$$

- If a letter is used odd times, then the EGF is

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = 0.5(e^x - e^{-x}).$$

Example: Determine the number of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times, and 5 occur at least once.

$$g(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots)^2 (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^3$$
$$= (\frac{e^x + e^{-x}}{2})^2 \cdot e^{3x} = \frac{1}{4}(e^{5x} + 2e^{3x} + e^x) = \frac{1}{4} \sum_{n=0}^{\infty} (5^n + 2 \cdot 3^n + 1) \frac{x^n}{n!}$$