Math 432 lec 13 Cayley formula, bipartite graph and matching

(1) How many trees do we have with vertex set $[n]$?

Cayley's formula: there are n^{n-2} spanning trees in a labelled K_n.

Proof: we build a bijection from $(n-2)$-list of $[n]$ to labelled spanning trees of K_n:

(Prufer code method) For each spanning tree, we repeatedly delete the least labelled leaf and put its neighbor in the list, until we have two vertices left. Then we have a $(n-2)$-list of $[n]$.

For each list $(t_1, t_2, \ldots, t_{n-2})$, we choose the least number $s_1 \in [n] - \{t_i : 1 \leq i \leq n-2\}$ and let it adjacent to t_1; then choose the least number $s_2 \in [n] - \{s_1, t_i : 2 \leq i \leq n-2\}$ and let it adjacent to t_2; repeat this, until we have exhausted the list. Now we have $n-2$ edges and join the two vertices in $[n] - \{s_1, s_2, \ldots, s_{n-2}\}$.

(2) A forest is a graph without any cycles. So a tree is a connected forest, and the components of a forest are trees. Let $w(F)$ be the number of components of F. Then a forest has at least $2w(F)$ leaves, and $n - w(F)$ edges.

(3) A bipartite graph G is a graph whose vertices can be partitioned into two parts (A and B) so that all the edges are between the two parts.

A tree is a bipartite graphs. (pf: start from a vertex, do a BFS search, and label the vertices in each level alternatively by 0 and 1.)

(4) Thm: A graph is bipartite if and only if it contains no odd cycle. (proof?)

(5) Def: The n-cube Q_n: the vertex set consists of all binary n-tuples, and two vertices are adjacent if the two n-tuples differ by exactly one coordinate. One may see Q_2 is 4-cycle.

Why is Q_n a bipartite graph for any n?

(6) Def: A *matching* is a collection of edges which share no endpoint. A *maximum matching* is a matching with largest size in the graph; a *maximal matching* is one which cannot be enlarged. A *perfect matching* is one covering all the vertices (thus contains $n/2$ edges).

(7) Can you show a graph and a maximal matching in it which is not a maximum matching?