Math 432 lec 14 Bipartite graph and matching

Def: A matching is a collection of edges which share no endpoint.

Def: A maximum matching is a matching with largest size in the graph; a maximal matching is one which cannot be enlarged. A perfect matching is one covering all the vertices (thus contains $n/2$ edges).

Can you show a graph and a maximal matching in it which is not a maximum matching?

Ex: there are $n!$ perfect matchings in $K_{n,n}$. There are $(2n)!/(2^n n!)$ perfect matchings in K_{2n}.

Def: An M-alternating path and an M-augmenting path. Symmetric difference of two matchings $M \Delta M'$.

Theorem (Berge): a matching M is maximum in G if and only if G has no M-augmenting path. (proof)

Theorem (P. Hall 1935) If G is a bipartite graph with bipartition X and Y, then G has a matching of X into Y if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof (of sufficiency): consider a maximum matching M from X into Y and let u be a vertex not covered by M. Then let $S \subseteq X$ and $T \subseteq Y$ be the vertices reachable by M-alternating paths from u. We show that $N(S) = T$ and $|T| = |S| - 1$, a contradiction.

Corollary: for $k > 0$, every k-regular bipartite graph has a perfect matching.