Math 432 lec 15 Matching Theory

0.1. **Solve Chinese Postman Problem.** We can construct a weighted complete graphs on the vertices of odd degrees in G, the weight on an edge uv is the length of the shortest u, v-path in G. Now we can find a minimum weighed matching in the complete graph. In G, we could add new edges between the endpoints of the matching, and find an Eulerian circuit in the resulting graphs, then replace the new edges with the shorted path between the endpoints to get a solution to the CPP.

0.2. **matching and vertex cover.** A vertex cover is a set S of vertices such that S contains at least one endpoint of every edge of G.

Ex: matching and vertex cover of C_5.

Theorem (Kónig, Egerváry 1931) If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof: a vertex cover takes at least one vertex from each edge of a matching. So we just need to show a matching of the size of a minimum vertex cover. This can be done using Hall's theorem (by considering two sub-bipartite graphs).

0.3. **Matching in general graphs.** Def: let $o(G - S)$ be the number of odd components of $G - S$. Define the deficiency of a graph to be $def(G) = \max_{S \subseteq G}\{o(G - S) - |S|\}$.

General Matching: Tutte's 1-Factor Theorem: a graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for all $S \subseteq V(G)$.

Theorem (Petersen) every bridgeless 3-regular graph has a 1-factor.

Pf: let $G_1, \ldots, G_k$ be the odd components of $G - S$, and let m_i be the number of edges with one endpoint in G_i (the other is on S). Then m_i is odd, and $m_i > 1$, so $m_i \geq 3$.
Now count the total degree of vertices in S: we have $3|S| \geq \sum_{v \in S} d(v) \geq \sum_i m_i \geq 3k = 3o(G-S)$. By Tutte's theorem, we have a perfect matching in G.

Cor: every bridgeless 3-regular graph has a 2-facto.