Math 432 lec 17 Menger Theorem

Expansion Lemma: if G is k-connected, and G' is obtained by adding a new vertex y adjacent to at least k vertices of G, then G' is also k-connected.

Subdivision Lemma: A subdivision of a 2-connected graph is still 2-connected.

Theorem: Let $n(G) \geq 3$. Then G is 2-connected if and only if
(a) for any pair u, v, there are two internally disjoint u, v-paths.
(b) for all u, v, there is a cycle through u, v.
(c) for any edges uv, xy, there is a cycle though uv, xy.

Theorem (ear-decomposition) A graph is 2-connected if and only if it has a ear-decomposition. In addition, any cycle can be taken as the initial cycle of some ear-decompositio.

Menger's Theorem: a graph is k-connected if and only if there are k internally disjoint paths between any pair of vertices.
Proof: induction on $d(u, v)$.

Menger's Theorem (local version): the maximum number of internally disjoint paths between x and y equals to the minimum number of vertices in a vertex cut separating x and y. ($\kappa(x, y) = \lambda(x, y)$) (Proof: use induction on n and consider two cases depending on whether a x, y-cut contains $N(x)$ or $N(y)$.)