Math 432 lec 19-20 Vertex coloring

A proper k-coloring of G is a mapping $c : V(G) \to [k]$ so that $c(u) \neq c(v)$ if $uv \in E(G)$. The chromatic number $\chi(G)$ of G is defined to be $\chi(G) = \min\{k : G \text{ has a proper } k\text{-coloring}\}$.

In some scheduling problems, we can construct a conflict graph, and a proper coloring provides an actual solution to the problem.

Prop: G is a bipartite graph if and only of $\chi(G) \leq 2$.

To show $\chi(G) = k$, we need two parts: we can color the graph with k colors, and no one can color the graph with fewer than k colors. That is, we need to show k is an upper bound and a lower bound of $\chi(G)$.

Upper bounds:

Prop: $\chi(G) \leq \Delta + 1$. (pf: We use a greedy algorithm to prove it.)

Brook's Theorem: $\chi(G) \leq \Delta + 1$ if G is not an odd cycle or complete graph.
Proof: We find a special ordering of the vertices: $u_1, u_2, \ldots, v_n$ so that $u_1u_n, u_2u_n \in E(G)$ and $u_1u_2 \notin E(G)$, and every u_i has at least one neighbor u_j with $j > i$. Then color the vertices greedily.
Find three vertices u, v, w so that $uv, uw \in E(G)$ and $vw \notin E(G)$, and $G - \{v, w\}$ is connected.

A graph G is d-degenerate if every subgraph of G has a vertex with degree at most d. Note that a d-degenerate graph with n-vertices has at most dn edges.

Them: if G is d-degenerate, then $\chi(G) \leq d + 1$. (pf: greedy coloring)

Lower bounds:

Prop: $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the clique number of G.

Myciesky's Construction: there are graphs with $\omega(G) = 2$ and arbitrary large $\chi(G)$.

Erdos proved that there are graphs with arbitrary high girth and arbitrary high chromatic number.

From $e(G) \leq 3n - 6$, a planar graph G is 5-degenerate, i.e., every subgraph contains a vertex with degree at most 5. So $\chi(G) \leq 6$.

Theorem: $\chi(G) \leq 5$ for any planar graph G. (Proof: use Kemp chain)
Theorem: $\chi(G) \leq 4$ for every planar graph G. (describe briefly the discharing method)