On strong edge-coloring of graphs with maximum degree 4

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Abstract

The strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is the least number of colors needed to edge-color G properly so that every path of length 3 uses three different colors. In this paper, we prove that if G is a graph with $\Delta(G) = 4$ and maximum average degree less than $\frac{61}{18}$ (resp. $\frac{7}{2}$, $\frac{18}{5}$, $\frac{15}{4}$, $\frac{51}{13}$), then $\chi'_s(G) \leq 16$ (resp.17, 18, 19, 20), which improves the results of Bensmail, Bonamy, and Hocquard (2015).

1 Introduction

A *strong edge-coloring* of a graph G is a proper edge-coloring of G such that the edges of any path of length 3 use three different colors. It follows that each color class of a strong edge-coloring is an induced matching. The strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is the smallest integer k such that G can be strongly edge-colored with k colors. The concept of strong edge-coloring was introduced by Fouquet and Jolivet in [8, 9] and can be used to model conflict-free channel assignment in radio networks in [16, 17].

In 1985, Erdős and Nešetřil proposed the following interesting conjecture.

Conjecture 1.1 ([7]) *For a graph G with maximum degree Δ,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

When $\Delta \leq 3$, Conjecture 1.1 has been verified by Andersen [1], and independently by Horák, Qing, and Trotter [13]. When Δ is sufficiently large, Molloy and Reed in [15] proved that $\chi'_s(G) \leq 1.998\Delta(G)^2$, using probabilistic techniques. This bound is improved to $1.93\Delta^2$ by Bruhn and Joos [3], and very recently, is further improved to $1.835\Delta^2$ by Bonamy, Perrett, and Postle [4].

The maximum average degree of a graph G, $mad(G)$, is defined to be the maximum average degree over all subgraphs of G. Hocquard *et al.* [11, 12] and DeOrsey *et al.* [6] studied the strong chromatic index of subcubic graphs with bounded maximum average degree.

We study graphs with maximum degree 4, which are conjectured to be colorable with at most 20 colors in Conjecture 1.1. Cranston [5] showed that 22 colours suffice, which is improved to

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21 colours very recently by Huang, Santana and the third author [14]. However, it is still not clear if 20 colours suffice even if the minimum degree of such graphs is 3. Bensmail, Bonamy, and Hocquard [2] studied the strong chromic index of graphs with maximum degree four and bounded maximum average degrees.

Theorem 1.2 (Bensmail, Bonamy, and Hocquard [2]) *For every graph G with $\Delta = 4$,*

(1) If $mad(G) < \frac{16}{5}$, then $\chi'_s(G) \leq 16$.

(2) If $mad(G) < \frac{10}{3}$, then $\chi'_s(G) \leq 17$.

(3) If $mad(G) < \frac{17}{5}$, then $\chi'_s(G) \leq 18$.

(4) If $mad(G) < \frac{18}{5}$, then $\chi'_s(G) \leq 19$.

(5) If $mad(G) < \frac{19}{5}$, then $\chi'_s(G) \leq 20$.

In this paper, we improve the results from [2] as follows.

Theorem 1.3 *For every graph G with $\Delta = 4$, each of the following holds.*

(1) If $mad(G) < \frac{61}{18}$, then $\chi'_s(G) \leq 16$.

(2) If $mad(G) < \frac{7}{2}$, then $\chi'_s(G) \leq 17$.

(3) If $mad(G) < \frac{18}{5}$, then $\chi'_s(G) \leq 18$.

(4) If $mad(G) < \frac{15}{4}$, then $\chi'_s(G) \leq 19$.

(5) If $mad(G) < \frac{51}{13}$, then $\chi'_s(G) \leq 20$.

From the proof of Theorem 1.3(5), we obtain the following corollary, which implies Conjecture 1.1 is true in some spacial cases.

Corollary 1.4 *For every graph G with $\Delta = 4$, if there are two 3-vertices whose distance is at most 4, then $\chi'_s(G) \leq 20$.*

We end this section with notation and terminology. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $d_G(v)$ denote the degree of a vertex v in a graph G. We use V, E and $d(v)$ for $V(G)$, $E(G)$ and $d_G(v)$, respectively, if it is understood from the context. Denote by $d(u, v)$ the distance between vertices u and v of G. A vertex is a *k-vertex* (*k^--vertex*) if it is of degree k (at most k). Similarly, a neighbor of a vertex v is a *k-neighbor* of v if it is of degree k. A 4-vertex is *special* if it is adjacent to a 2-vertex. A 3-vertex is a *3_k-vertex* if it is adjacent to k 3-vertices, where $k = 0, 1, 2$. A *4_k-vertex* is a 4-vertex adjacent to exactly k 3-vertices. Denote by $N(v)$ the neighborhood of the vertex v, let $N_i(v) = \{u \in V(G) : d(u, v) = i\}$ for $i \geq 1$. For simplicity, $N_0(v) = \{v\}$ and $N_1(v) = N(v)$. Let $L_i(v) = \cup_{j=0}^{i} N_j(v)$ and $D_3(G) = \{v \in V(G) : d(v) = 3\}$. For a graph $G = (V, E)$ and $E' \subseteq E$, G has a *partial edge-coloring* if $G[E']$ has a strong edge-coloring, where $G[E']$ is the graph with vertex set V and edge set E'.

In the proof of Theorem 1.3, the well known result of Hall [10] is applied in terms of systems of distinct representatives.

Theorem 1.5 ([10]) *Let $A_1, \ldots, A_n$ be n subsets of a set U. A system of distinct representatives of $\{A_1, \ldots, A_n\}$ exists if and only if for all $k, 1 \leq k \leq n$ and every subcollection of size k, $\{A_{i_1}, \ldots, A_{i_k}\}$, we have $|A_{i_1} \cup \ldots \cup A_{i_k}| \geq k$.*

2 Proof of Theorem 1.3

Let H be a counterexample to Theorem 1.3 with $|V(H)| + |E(H)|$ minimized. That is, for some

$$(m, k) \in \{(\frac{61}{18}, 16),\ (\frac{7}{2}, 17),\ (\frac{18}{5}, 18),\ (\frac{15}{4}, 19),\ (\frac{51}{13}, 20)\}$$

we have $mad(H) < m$ and $\chi'_s(H) > k$.

By the minimality of H, $\chi'_s(H - e) \leq k$ for each $e \in E(H)$, and we may assume that H is connected. Denote by $[k] = \{1, 2, \ldots, k\}$ the set of colors. If $e = uv$ is an uncolored edge in a partial coloring of H, then let $L_H(e)$ be the set of colors that is used on the edges incident to a vertex in $N_H(u) \cup N_H(v)$, and let $L'_H(e) = [k] \setminus L_H(e)$. We write $L(e)$ and $L'(e)$ for $L_H(e)$ and $L'_H(e)$, respectively, if it is clear from the context. We now establish some properties of H.

Lemma 2.1 *Let x be a vertex of H with $d(x) = d$. If the edges incident to x can be ordered as $xy_1, xy_2, \ldots, xy_d$ such that in a partial k-coloring of $H - x$, $|L(xy_i)| \leq k - i$, then the partial coloring can be extended to H. In particular,*

(a) There is no 1-vertex in H, an if $k \geq 17$, then there is no 2-vertex in H.

(b) Each 2-vertex x in H has two 4-neighbors, each of which is adjacent to three 4-vertices.

(c) If $d(x) = 3$ and $k \geq 16, 17, 19$, then x is adjacent to at least one, two, and three 4-vertices, respectively.

(d) If $d(x) = 4$ and if $k \geq 18, 19, 20$, then x is adjacent to at most three, two and one 3-vertices, respectively.

Proof. We color $xy_d, xy_{d1}, \ldots, xy_1$ in order and obtain a strong-edge coloring of H. For the "in particular" part, let x be $d(x) = d$ and the neighbors of x are $y_1, y_2, \ldots, y_d$ with $d(y_1) \geq d(y_2) \leq \ldots \geq d(y_d)$. Then in each case, $H - x$ has a strong k-edge-coloring.

(a) When $d(x) = 1$, $|L(xy)| \geq k - 12 \geq 4$, so xy can be colored. When $d(x) = 2$, then $|L(xy_1)|, |L(xy_2)| \geq k - 15 \geq 2$ if $k \geq 17$, so there is no 2-vertex if $k \geq 17$.

(b) As $d(x) = 2$, $|L(xy_1)|, |L(xy_2)| \geq k - 15 \geq 1$, with $|L(xy_1)| = |L(xy_2)| = 1$ only if both y_1 and y_2 are 4-vertices and adjacent to three 4-neighbors. So if y_1 or y_2 is not a 4-vertex or one of them is not adjacent to three 4-neighbors, we can color xy_1 and xy_2.

(c) Note that $d(x) = 3$ and $d(y_1) \geq d(y_2) \geq d(y_3)$. If x has three 3-neighbors and $k \geq 16$, then $|L(xy_i)| \leq 12 \leq k - 4$; if x has two 3-neighbors and $k \geq 17$, then $|L(xy_1)| \leq 16 \leq k - 1$ and $|L(xy_2)|, |L(xy_3)| \leq 13 \leq k - 4$; if x has one 3-neighbors and $k \geq 19$, then $|L(xy_1)|, |L(xy_2)| \leq 17 \leq k - 2$ and $|L(xy_3)| \leq 14 \leq k - 5$. So by the main statement, the coloring of $H - x$ can be extended to H in each of the cases.

(d) Note that $d(x) = 4$ and $d(y_1) \geq d(y_2) \geq d(y_3) \geq d(xy_4)$. If x has four 3-neighbors and $k \geq 18$, then $|L(xy_i)| \leq 14 \leq k-4$; if x has three 3-neighbors and $k \geq 19$, then $|L(xy_1)| \leq 18 \leq k-1$ and $|L(xy_i)| \leq 15 \leq k - 4$ for $i \in \{2, 3, 4\}$. So by the main statement, the coloring of $H - x$ can be extended to H in each of the cases. When $k \geq 20$ and x has two 3-neighbors, we uncolor y_4w, where $w \neq x$ is a neighbor of y_4. Then $|L'(xy_1)|, |L'(xy_1)| \geq 2$ and $|L'(xy_3)|, |L'(xy_4)| \geq 5$ and $|L'(y_4w)| \geq 4$. So we can color $xy_1, xy_2, y_4w, xy_3, xy_4$ in the order and obtain a coloring of H. ■

Let the initial charge of $x \in V(H)$ be $\omega(x) = d(x) - m$. It follows from the hypothesis that $\sum_{x \in V(H)} \omega(x) < 0$. We redistribute the weights using the following discharging rules:

(R1) When $k = 16$, each 4-vertex v gives $4 - m$ to its unique 2-neighbor if it has one. Otherwise, it gives $\frac{3m-10}{6}$ to the 2-vertices in $L_2(v)$. It gives $m - 3$ to each 3_2-neighbor, $\frac{m-3}{2}$ to each 3_1-neighbor, and $\frac{m-3}{3}$ to each 3_0-neighbor.

(R2) When $k \geq 17$, each 4-vertex u gives $\frac{4-m}{l}$ to each of the l 3-vertices in $L_{i+1}(u) \cap D_3(G)$ when $L_i(u) \cap D_3(G)$ is empty, where $i \geq 0$.

For each vertex $x \in V(H)$, let $\omega^*(x)$ be the final weight of x after the discharging process. If each vertex $x \in V(H)$ has $\omega^*(x) \geq 0$, then

$$0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0.$$

This is a contradiction. So there must be some vertex, say $x_0 \in V(H)$, with $\omega^*(x_0) < 0$.

Lemma 2.2 *If $k \geq 17$, then x_0 is a 3-vertex. If $k = 16$, then x_0 is a 4-vertex.*

Proof. If $k \geq 17$, then there is no 2-vertex by Lemma 2.1 (a). By (R2), $\omega^*(x) = 0$ if $d(x) = 4$. So, x_0 is a 3-vertex.

Let $k = 16$. By Lemmas 2.1 (a) and 2.1 (b), each 2-vertex x is adjacent to two 4-vertices in $N(x)$ and adjacent to six 4-vertices in $N_2(x) \backslash N(x)$. By (R1), $\omega^*(x) = 2 - \frac{61}{18} + 2(4 - \frac{61}{18}) + 6 \cdot (3 \cdot \frac{61}{18} - 10)/6 = 0$. Assume that x_0 is a 3-vertex. If x_0 is a 3_2-vertex, by (R1), $\omega(x_0) = 3 - \frac{61}{18} + \frac{61}{18} - 3 = 0$, a contradiction; if x_0 is a 3_1-vertex, then by (R1), $\omega(x_0) = 3 - \frac{61}{18} + 2 \cdot (\frac{61}{18} - 3)/2 = 0$, a contradiction; if x_0 is a 3_0-vertex, then by (R1) $\omega(x_0) = 3 - \frac{61}{18} + 3 \cdot (\frac{61}{18} - 3)/3 = 0$, a contradiction; Thus, x_0 is not a 3-vertex. So, x_0 is a 4-vertex. ■

2.1 Case 1: $(m, k) = (\frac{61}{18}, 16)$

Lemma 2.3 *If v is a 3_2-vertex, then its 4-neighbor is adjacent to three 4-vertices.*

Proof. Suppose to the contrary that a 3-vertex v is adjacent to two 3-vertices u and w and a 4-vertex t that is adjacent to a 3-vertex t_1. By the minimality of H, $H' = H - v$ has a strong edge-coloring with at most sixteen colors. Observe that $|L'(uv)| \geq 3$, $|L'(vw)| \geq 3$ and $|L'(vt)| \geq 1$. Thus, we color vt, uv and vw in turn to obtain a strong edge-coloring of H, a contradiction. ■

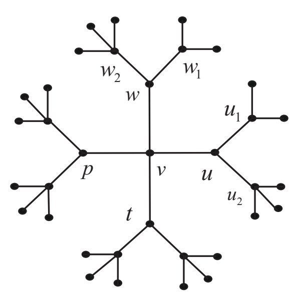

Figure 1: A 4-vertex v adjacent to four 3-vertices

Lemma 2.4 *A 4_4-vertex v is adjacent to at most one 3_1-vertex.*

Proof. Suppose otherwise that there exists a 4_4-vertex v adjacent to two 3_1-vertices w and u. Let $d(u_1) = d(w_1) = 3$. We use notations in Figure 1. By the minimality of H, $H' = H - v$ has a strong edge-coloring with at most 16 colors.

We claim that $w_1 \neq u_1$. For otherwise, $|L'(uv)| \geq 4$, $|L'(vw)| \geq 4$, $|L'(vp)| \geq 2$ and $|L'(vt)| \geq 2$. Thus, we color vt, vp, vw and vu in turn to obtain a strong edge-coloring of H, a contradiction.

We also claim that $u_1w_1 \notin E(H)$. Suppose otherwise. We uncolor edges uu_1 and ww_1. Then $|L'(uv)| \geq 5$, $|L'(vw)| \geq 5$, $|L'(vt)| \geq 4$, $|L'(vp)| \geq 4$, $|L'(uu_1)| \geq 6$, $|L'(ww_1)| \geq 6$. Then we color edges vt, vp, vw, uv, uu_1 and ww_1 in turn to obtain a strong edge-coloring of H, a contradiction.

Now, we uncolor edges uu_1 and ww_1. Then $|L'(uv)| \geq 5$, $|L'(vw)| \geq 5$, $|L'(vt)| \geq 4$, $|L'(vp)| \geq 4$, $|L'(uu_1)| \geq 4$, $|L'(ww_1)| \geq 4$. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, then we color edges uu_1, ww_1 with a same color and then color vt, vp, vw and uv in turn. If $L'(uu_1) \cap L'(ww_1) = \emptyset$, then $|L'(uu_1) \cup L'(ww_1)| \geq 8$. Let $T = \{uv, vw, vt, vp, uu_1, ww_1\}$, for any $S \subseteq T$, we have $|\cup_{e \in S} L^*(e)| \geq |S|$. By Theorem 1.5, we can assign a distinct color to each uncolored edge. Thus, we obtain a strong edge-coloring of H, a contradiction. ■

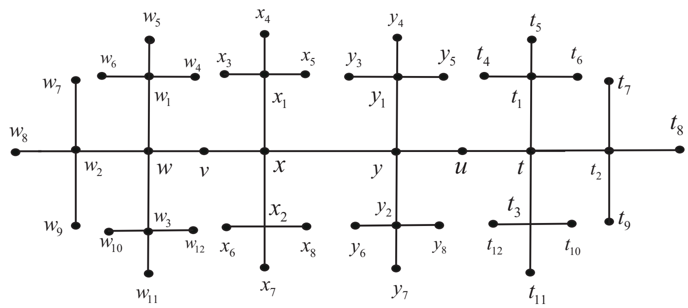

Figure 2: The distance between two 2-vertices v and u is 3

Lemma 2.5 *The distance between two 2-vertices is at least 4.*

Proof. By Lemma 2.1 (b), the distance between every two 2-vertices is at least 3. Suppose otherwise that there exist two 2-vertices u and v with $d(u, v) = 3$. We shall use the notations in Figure 2. By the minimality of H, $H' = H - \{v, u\}$ has a strong edge-coloring with at most sixteen colors. One can observe that $|L'(wv)| \geq 1$, $|L'(vx)| \geq 2$, $|L'(ut)| \geq 1$, $|L'(yu)| \geq 2$.

We first claim that $|L'(vx)| = |L'(uy)| = 2$. By symmetry, suppose otherwise that $|L'(vx)| \geq 3$. In this case, we can color wv, ut, uy and vx in turn and we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

Next, we claim that $|L'(wv)| = |L'(ut)| = 1$. By symmetry, suppose otherwise that $|L'(wv)| \geq 2$. Thus, we can color ut, uy, vx and wv in turn and we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

Finally, we claim that $L'(wv) \subseteq L'(vx)$ and $L'(ut) \subseteq L'(uy)$. By symmetry, suppose otherwise that if $L'(wv) \nsubseteq L'(vx)$. In this case, we can color ut, uy, vx and wv in turn and we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

We distinguish the following two cases:

Case 1. $L'(vx) \neq L'(uy)$.

If $L'(vx) \cap L'(uy) = \emptyset$, then we can color vw, ut, vx and uy in turn and we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

Thus, we assume that $L'(vx) \cap L'(uy) \neq \emptyset$. Since $L'(vx) \neq L'(uy)$, we assume, without loss of generality, that $L'(vx) = \{1, 2\}$ and $L'(uy) = \{1, 3\}$. If $L'(wv) = L'(ut) = \{1\}$, we can color wv and ut with 1, and color vx and uy with 2 and 3, respectively. It follows that we obtain a desired strong edge-coloring with sixteen colors, a contradiction. So, we assume that $L'(wv) \neq L'(ut)$. By symmetry we may assume that either $L'(wv) = \{1\}$ and $L'(ut) = \{3\}$ or $L'(wv) = \{2\}$ and

$L'(ut) = \{3\}$. In the former case, we can color wv and yu with 1, and color vx and ut with 2 and 3, respectively. So, we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

In the latter case, we assume, without loss of generality, that $c(xy) = 4$. Note that $4 \notin \{c(ww_1), c(ww_2), c(ww_3), c(w_1w_4), c(w_1w_5), c(w_1w_6), c(w_2w_7), c(w_2w_8), c(w_2w_9), c(w_3w_{10}), c(w_3w_{11}), c(w_3w_{12})\}$, for otherwise we obtain $|L'(wv)| \geq 2$, contrary to our claim that $|L'(wv)| = 1$. Similarly, $4 \notin \{c(tt_1), c(tt_2), c(tt_3), c(t_1t_4), c(t_1t_5), c(t_1t_6), c(t_2t_7), c(t_2t_8), c(t_2t_9), c(t_3t_{10}), c(t_3t_{11}), c(t_3t_{12})\}$. Since $L'(vx) = \{1, 2\}$ and $L'(uy) = \{1, 3\}$, $1 \notin \{c(xx_1), c(xx_2), c(x_1x_3), c(x_1x_4), c(x_1x_5), c(x_2x_6), c(x_2x_7), c(x_2x_8), c(yy_1), c(yy_2), c(y_1y_3), c(y_1y_4), c(y_1y_5), c(y_2y_6), c(y_2y_7), c(y_2y_8)\}$. Thus, we can recolor xy with 1 and color wv, ut with the same color 4, color vx, yu with with 2 and 3, respectively, and we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

Case 2. $L'(vx) = L'(uy)$.

In this case, we assume, without loss of generality, that $L'(vx) = L'(uy) = \{1, 2\}$. By symmetry, we assume that either $L'(wv) = \{1\}$ and $L'(ut) = \{2\}$ or $L'(wv) = L'(ut) = \{1\}$. In the former case, we can color wv, uy with the same color 1 and color vx, ut with the same color 2. So, we obtain a desired strong edge-coloring with sixteen colors, a contradiction.

In the latter case, we assume, without loss of generality, that $c(xy) = 3$. Note that $3 \notin \{c(ww_1), c(ww_2), c(ww_3), c(w_1w_4), c(w_1w_5), c(w_1w_6), c(w_2w_7), c(w_2w_8), c(w_2w_9), c(w_3w_{10}), c(w_3w_{11}), c(w_3w_{12})\}$, for otherwise, we obtain that $|L'(wv)| \geq 2$, contrary to our claim that $|L'(wv)| = 1$. Similarly, $3 \notin \{c(tt_1), c(tt_2), c(tt_3), c(t_1t_4), c(t_1t_5), c(t_1t_6), c(t_2t_7), c(t_2t_8), c(t_2t_9), c(t_3t_{10}), c(t_3t_{11}), c(t_3t_{12})\}$. Since $L'(vx) = \{1, 2\} = L'(uy) = \{1, 2\}$, $2 \notin \{c(xx_1), c(xx_2), c(x_1x_3), c(x_1x_4), c(x_1x_5), c(x_2x_6), c(x_2x_7), c(x_2x_8), c(yy_1), c(yy_2), c(y_1y_3), c(y_1y_4), c(y_1y_5), c(y_2y_6), c(y_2y_7), c(y_2y_8)\}$. Thus, we can recolor xy with 2 and color both wv and uy with 3, color both vx and ut with 1. Therefore, we obtain a desired strong edge-coloring with sixteen colors, a contradiction. ■

Consider the final charge of x_0. By Lemma 2.2, x_0 is a 4-vertex.

If x_0 is adjacent to a 2-vertex, then by Lemma 2.1 (b), the other three neighbors are all 4-vertices. By (R1), $\omega^*(x_0) \geq 4 - \frac{61}{18} - (4 - \frac{61}{18}) = 0$, a contradiction. Thus, x_0 has no 2-neighbor. By Lemma 2.5, each 4-neighbor of x_0 (if any) is adjacent to at most one 2-vertex.

If x_0 is adjacent to a 3_2-vertex, then by Lemma 2.3, the other three neighbors are 4-vertices. By (R1), $\omega^*(x_0) \geq 4 - \frac{61}{18} - (\frac{61}{18} - 3) - 3 \cdot (3 \cdot \frac{61}{18} - 10)/6 = \frac{5}{36} > 0$, a contradiction. Thus, x_0 is not adjacent to any 3_2-neighbor. Assume that x_0 is adjacent to a 3_1-vertex. If x_0 is a 4_4-vertex, then by Lemma 2.4, x_0 is adjacent to at most one 3_1-vertex. By (R1), $\omega^*(x_0) \geq 4 - \frac{61}{18} - (\frac{61}{18} - 3)/2 - 3 \cdot (\frac{61}{18} - 3)/3 = \frac{1}{36} > 0$. If x_0 is not a 4_4-vertex, then by (R1), $\omega^*(x_0) \geq 4 - \frac{61}{18} - 3 \cdot (\frac{61}{18} - 3)/2 - (3 \cdot \frac{61}{18} - 10)/6 = (61 - 18 \cdot \frac{61}{18})/6 = 0$, a contradiction. Thus, x_0 is adjacent to only 3_0-neighbors or 4-vertices. By (R1), $\omega^*(x_0) \geq 4 - \frac{61}{18} - 4 \cdot (\frac{61}{18} - 3)/3 = (24 - 7 \cdot \frac{61}{18})/3 > 0$, contrary to the assumption that $\omega^*(x_0) < 0$.

2.2 Case 2: $(m, k) = (7/2, 17)$

Lemma 2.6 *H does not contain the following three configurations:*

(1) A 3_1-vertex v adjacent to a 4_3-vertex u (see Figure 3).

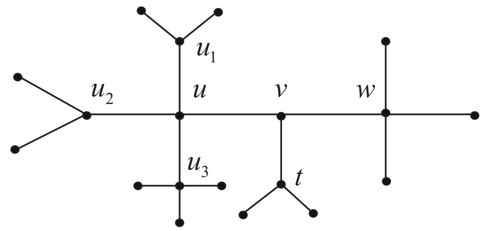

Figure 3: A 3_1-vertex v adjacent to a 4_3-vertex u

(2) A 3_0-vertex v adjacent to two 4_4-vertices u, w and one 4_3-vertex t (see Figure 4).
(3) A 3_0-vertex v adjacent to one 4_4-vertex u and two 4_3-vertices w, t (see Figure 5).

Proof. (1) Suppose otherwise that there exists a 3_1-vertex v that is adjacent to a 4_3-vertex u. Let t, u_1 and u_2 be 3-vertices and let w and u_3 be 4-vertices. we use the notations in Figure 3. By minimality of H, $H' = H - \{u, v\}$ has a strong edge-coloring with at most seventeen colors. Observe that $|L'(uv)| \geq 5$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 6$, $|L'(uu_1)| \geq 4$, $|L'(uu_2)| \geq 4$ and $|L'(uu_3)| \geq 1$. Thus, we color uu_3, vw, uu_1, uu_2,uv and vt in turn and obtain a desired strong edge-coloring with seventeen colors, a contradiction.

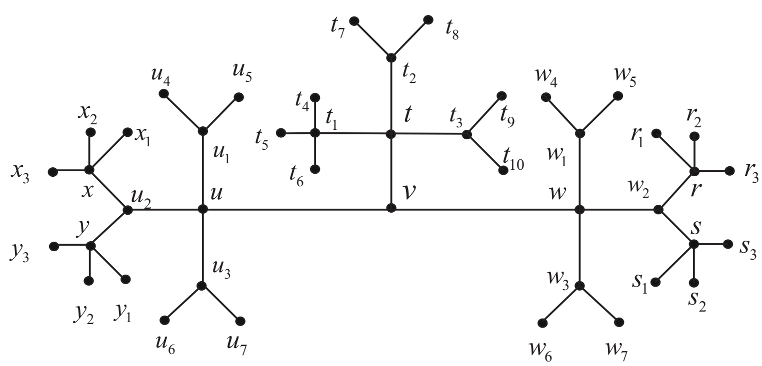

Figure 4: A 3_0-vertex v adjacent to two 4_4-vertices and one 4_3-vertex

(2) Suppose otherwise that there exists a 3_0-vertex v adjacent to two 4_4-vertices u, w and one 4_3-vertex t. We shall use the notations Figure 4. Let $H' = H - \{v\}$. By the minimality of H, H' has a strong edge-coloring with at most seventeen colors. Observe that $|L'(uv)| \geq 2$, $|L'(vw)| \geq 2$, $|L'(vt)| \geq 1$. Note that there are 3 uncolored edges. If we can assign a distinct color to uncolored edge, then we obtain a desired strong edge-coloring with seventeen colors, a contradiction.

Thus, assume that we cannot assign three distinct colors to these three uncolored edges. By Theorem 1.5, $L'(vt) \subseteq L'(uv) = L'(vw)$ and $|L'(uv)| = 2$. Without loss of generality, we consider the following two cases.

Case 1. $L'(vt) = \{1\}$, $L'(uv) = L'(vw) = \{1, 2\}$.

Since $L'(vt) = \{1\}$, $c(tt_1), c(tt_2), c(tt_3), c(uu_1), c(uu_2), c(uu_3), c(ww_1), c(ww_2)$ and $c(ww_3)$ are distinct. Suppose otherwise. We obtain $|L'(uv)| \geq 3$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 2$. In this case, we can color vt, uv and vw and obtain a desired strong edge-coloring with seventeen colors, a contradiction. Thus, since $L'(vt) = \{1\}$ and $L'(uv) = L'(vw) = \{1, 2\}$, we may assume, without loss of generality, that $c(tt_1) = 3$, $c(tt_2) = 4$, $c(tt_3) = 5$, $c(uu_1) = 6$, $c(uu_2) = 7$, $c(uu_3) = 8$, $c(ww_1) = 9$, $c(ww_2) = 10$, $c(ww_3) = 11$, $c(t_1t_4) = 12$, $c(t_1t_5) = 13$, $c(t_1t_6) = 14$, $c(t_2t_7) = 15$, $c(t_2t_8) = 16$, $c(t_3t_9) = 17$, $c(t_3t_{10}) = 2$, $c(u_1u_4) = 12$, $c(u_1u_5) = 13$, $c(u_2x) = 14$, $c(u_2y) = 15$, $c(u_3u_6) = 16$,

$c(u_3u_7) = 17$, $c(w_1w_4) = 12$, $c(w_1w_5) = 13$, $c(w_2r) = 14$, $c(w_2s) = 15$, $c(w_3w_6) = 16$, $c(w_3w_7) = 17$. This implies that $\{c(xx_1), c(xx_2), c(xx_3), c(yy_1), c(yy_2), c(yy_3)\} = \{3, 4, 5, 9, 10, 11\}$. Suppose otherwise. We can pick a color $\alpha \in \{3, 4, 5, 9, 10, 11\} \setminus \{c(xx_1), c(xx_2), c(xx_3), c(yy_1), c(yy_2), c(yy_3)\}$, recolor uu_2 with α and then we can color uv with 7, vw with 2, vt with 1. So, we obtain a desired strong edge-coloring with seventeen colors, a contradiction. Similarly, we can prove that $\{c(rr_1), c(rr_2), c(rr_3), c(ss_1), c(ss_2), c(ss_3)\} = \{3, 4, 5, 6, 7, 8\}$. Therefore, we can recolor both uu_2 and ww_2 with 2, then color uv with 7, vw with 10, vt with 1. Thus, we obtain a desired strong edge-coloring with seventeen colors, a contradiction.

Case 2. $L'(vt) = L'(uv) = L'(vw) = \{1, 2\}$.

Since $L'(vt) = L'(uv) = L'(vw) = \{1, 2\}$, $c(tt_1), c(tt_2), c(tt_3), c(uu_1), c(uu_2), c(uu_3), c(ww_1)$, $c(ww_2)$ and $c(ww_3)$ are distinct. We assume, without loss of generality, that $c(uu_1) = 3$, $c(uu_2) = 4$, $c(uu_3) = 5$, $c(ww_1) = 6$, $c(ww_2) = 7$, $c(ww_3) = 8$, $c(tt_1) = 9$, $c(tt_2) = 10$, $c(tt_3) = 11$, $c(u_1u_4) = 12$, $c(u_1u_5) = 13$, $c(u_2x) = 14$, $c(u_2y) = 15$, $c(u_3u_6) = 16$, $c(u_3u_7) = 17$, $c(w_1w_4) = 12$, $c(w_1w_5) = 13$, $c(w_2r) = 14$, $c(w_2s) = 15$, $c(w_3w_6) = 16$, $c(w_3w_7) = 17$, $c(t_1t_4) = 12$, $c(t_1t_5) = 13$, $c(t_1t_6) = 14$, $c(t_2t_7) = 15$, $c(t_2t_8) = 16$, $c(t_3t_9) = 17$. Since $L'(vt) = \{1, 2\}$, $c(t_3t_{10}) \in \{3, 4, 5, 6, 7, 8, 12, 13, 14, 15, 16\}$. This implies that $\{c(xx_1), c(xx_2), c(xx_3), c(yy_1), c(yy_2), c(yy_3)\} = \{6, 7, 8, 9, 10, 11\}$, for otherwise we can pick a color $\alpha \in \{6, 7, 8, 9, 10, 11\} \setminus \{c(xx_1), c(xx_2), c(xx_3), c(yy_1), c(yy_2), c(yy_3)\}$ and recolor uu_2 with α, then we can color uv with 4, vw with 2, vt with 1. So, we obtain a desired strong edge-coloring with seventeen colors, a contradiction. Similarly, $\{c(rr_1), c(rr_2), c(rr_3), c(ss_1), c(ss_2), c(ss_3)\} = \{3, 4, 5, 9, 10, 11\}$. Therefore, we can recolor both uu_2 and ww_2 with 2, then we can color uv with 4, vw with 7, vt with 1. Thus, we obtain a desired strong edge-coloring with seventeen colors, a contradiction.

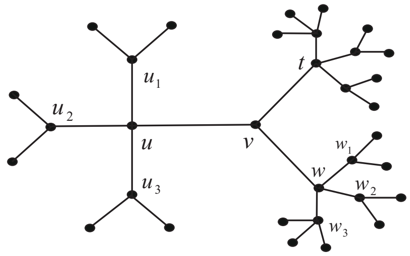

Figure 5: A 3_0-vertex v adjacent to one 4_4-vertex and two 4_3-vertices

(3) Suppose otherwise that a 3_0-vertex v is adjacent to one 4_4-vertex u and two 4_3-vertices w and t. Let each of u_1, u_2, u_3, w_1 and w_2 be a 3-vertex and w_3 is 4-vertex. We use the notations in Figure 5. By the minimality of H, $H' = H - v$ has a strong edge-coloring.

We claim that $u_i \neq w_j$, where $i = 1, 2, 3$ and $j = 1, 2$. For otherwise, $|L'(uv)| \geq 3$, $|L'(vt)| \geq 1$, $|L'(vw)| \geq 2$, we color vt, vw, and vu in turn to obtain a strong edge-coloring of H, a contradiction. By (1), a 3_1-vertex is not adjacent to a 4_3-vertex. Thus, $u_1w_1 \notin E(H)$.

Now, we erased the colors of edges uu_1, ww_1. Then $|L'(uv)| \geq 4$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 3$, $|L'(uu_1)| \geq 3$, $|L'(ww_1)| \geq 2$. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, then we color edges uu_1, ww_1 with the same color and then color vt, vw, uv in turn. Thus, we obtain a strong edge-coloring of H, a contradiction. If $L'(uu_1) \cap L'(ww_1) = \emptyset$, then $|L'(uu_1) \cup L'(ww_1)| \geq 5$. Let $T = \{uv, vw, vt, uu_1, ww_1\}$, for any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.5, we can assign a distinct color to uncolored edge. Thus, we obtain a strong edge-coloring of H, a contradiction. ■

Consider the final charge of x_0. By Lemma 2.2, x_0 is a 3-vertex. By Lemma 2.1 (c), x_0 is adjacent to at least two 4-vertices.

If x_0 is a 3_1-vertex, then x_0 is not adjacent to a 4_3-vertex by Lemma 2.6(1). Thus, $\omega^*(x_0) \geq 3-\frac{7}{2}+2\cdot(4-\frac{7}{2})/2 = 7-2\cdot\frac{7}{2} = 0$. Thus, we assume that x_0 is a 3_0-vertex. In this case, Lemma 2.6(2) implies that x_0 is adjacent to at most two 4_4-vertices. If x_0 is adjacent to two 4_4-vertices, then x_0 is not adjacent to 4_3-vertices by Lemma 2.6 (2). Thus, $\omega^*(x_0) \geq 3-\frac{7}{2}+2\cdot(4-\frac{7}{2})/4+(4-\frac{7}{2})/2 = 7-2\cdot\frac{7}{2} \geq 0$. If x_0 is adjacent to one 4_4-vertex, then x_0 is not adjacent to two 4_3-vertices by Lemma 2.6(3). Thus, $\omega^*(x_0) \geq 3-\frac{7}{2}+(4-\frac{7}{2})/4+(4-\frac{7}{2})/3+(4-\frac{7}{2})/2 = 22/3-\frac{25}{12}\cdot\frac{7}{2} = \frac{1}{24} > 0$. If v is not adjacent to 4_4-vertices, then $\omega^*(v) \geq 3-\frac{7}{2}+3\cdot(4-\frac{7}{2})/3 = 7-2\cdot\frac{7}{2} = 0$.

2.3 Case 3: $(m,k) = (\frac{18}{5}, 18)$

Lemma 2.7 *H does not contain the following two configurations:*
(1) A 3_1-vertex v adjacent to a 4_2-vertex u.
(2) A 3_0-vertex v adjacent to a 4_3-vertex u and a 4_2-vertex w (see Figure 6).

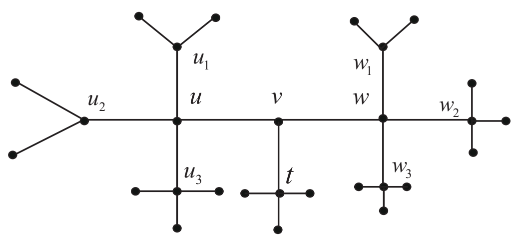

Figure 6: A 3_0-vertex v adjacent to a 4_3-vertex and a 4_2-vertex

Proof. (1) Suppose otherwise that a 3_1-vertex v is adjacent to a 4_2-vertex u. Let $N(v) = \{u, w, t\}$, where t is a 3-vertex and w is a 4-vertex. By the minimality of H, $H' = H - v$ has a strong edge-coloring with at most eighteen colors. It is easy to verify that $|L'(uv)| \geq 2$, $|L'(vt)| \geq 4$ and $|L'(vw)| \geq 1$. Thus, we color vw, uv, and vt in turn and we obtain a desired strong edge-coloring with eighteen colors, a contradiction.

(2) Suppose otherwise that a 3_0-vertex v is adjacent to a 4_3-vertex u and 4_2-vertex w. Let t, u_3, w_2 and w_3 be 4-vertices and let u_1, u_2 and w_1 be 3-vertices. We use the notations in Figure 6. By the minimality of H, $H' = H - v$ has a strong edge-coloring.

We claim that $w_1 \neq u_i$, where $i = 1, 2$. Suppose that $w_1 = u_1$. Uncolor uu_1, then $|L'(uv)| \geq 4$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 1$ and $|L'(uu_1)| \geq 4$. Thus, we can color vt, vw, uu_1 and uv in turn to obtain a strong edge-coloring of H, a contradiction.

By (1), a 3_1-vertex is not adjacent to a 4_2-vertex. Thus, $u_1w_1 \notin E(H)$.

Now, uncolor uu_1, ww_1, then $|L'(uv)| \geq 4$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 2$, $|L'(uu_1)| \geq 3$, $|L'(ww_1)| \geq 2$. If $L'(uu_1) \cap L'(ww_1) \neq \emptyset$, we color edges uu_1, ww_1 with the same color and color vt, vw, uv in turn to obtain a strong edge-coloring of H, a contradiction. Thus, we assume that $L'(uu_1) \cap L'(ww_1) = \emptyset$. Note that $|L'(uu_1) \cup L'(ww_1)| \geq 5$. Let $T = \{uv, vw, vt, uu_1, ww_1\}$, for any $S \subseteq T$, we have $|\cup_{e\in S} L'(e)| \geq |S|$. By Theorem 1.5, we can assign five distinct colors to uncolored edges. Thus, we obtain a strong edge-coloring with eighteen colors, a contradiction. ■

Consider the final charge of x_0. By Lemma 2.2, x_0 is a 3-vertex. By Lemma 2.1 (c), x_0 is adjacent to at least two 4-vertices. If x_0 is a 3_1-vertex, then x_0 is not adjacent to a 4_2-vertex by Lemma 2.7(1). Thus, by (R2), $\omega^*(x_0) \geq 3-\frac{18}{5}+(4-\frac{18}{5})\cdot 2 = 11-3\cdot\frac{18}{5} = \frac{1}{5} > 0$, a contradiction.

Thus, we assume that x_0 is a 3_0-vertex. By Lemma 2.1 (d), x_0 is not adjacent to a 4_4-vertex. If x_0 is adjacent to a 4_3-vertex, then x_0 is not adjacent to any 4_2-vertex by Lemma 2.7(2). This implies that $\omega^*(x_0) \geq 3-\frac{18}{5}+(4-\frac{18}{5})/3+(4-\frac{18}{5})\cdot 2 = \frac{28}{3}-9 = \frac{1}{3} > 0$, a contradiction.

If x_0 is not adjacent to any 4_3-vertex, then $\omega^*(x_0) \geq 3 - \frac{18}{5} + 3 \cdot (4 - \frac{18}{5})/2 = 9 - 5 \cdot (\frac{18}{5}/2) \geq 0$, a contradiction.

2.4 Case 4: $(m, k) = (\frac{15}{4}, 19)$

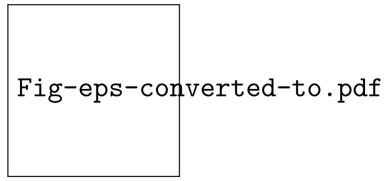

Figure 7: A 4_2-vertex

Lemma 2.8 *There is no 4_2-vertex.*

Proof. Suppose otherwise that u is a 4_2-vertex. Let u_1 and u_2 be 3-vertices and let u_3 and u_4 be 4-vertices. We shall use the notations in Figure 7. We first establish the following claims.

Claim 1. $\{u_{11}, u_{12}\} \cap \{u_{21}, u_{22}\} = \emptyset$.

Proof of Claim 1. Suppose otherwise that $u_{11} = u_{21}$ by symmetry. Let $H' = H - \{u, u_1, u_2\}$. By the minimality of H, H' has a strong edge-coloring with at most nineteen colors. In this case, one can see that $|L'(uu_3)| \geq 4, |L'(uu_4)| \geq 4, |L'(uu_1)| \geq 8, |L'(uu_2)| \geq 8, |L'(u_1u_{11})| \geq 8, |L'(u_1u_{12})| \geq 5, |L'(u_2u_{21})| \geq 8$ and $|L'(u_2u_{22})| \geq 5$. We can properly color $uu_3, uu_4, u_1u_{12}, u_2u_{22}, uu_1, uu_2, u_1u_{11}$ and u_2u_{21} in turn. Thus, we obtain a strong edge-coloring with nineteen colors, a contradiction. This proves Claim 1.

Claim 2. There is a pair of non adjacent vertices u_{1i} and u_{2j} for some $i, j \in \{1, 2\}$.

Proof of Claim 2. Suppose otherwise that for each $i, j \in \{1, 2\}$, $u_{1i}u_{2j} \in E(G)$. In this case, let $N(u_{1i}) = \{u_1, u_{21}, u_{22}, u'_{1i}\}$ for $i = 1, 2$ and $N(u_{2j}) = \{u_2, u_{11}, u_{12}, u'_{2j}\}$ for $j = 1, 2$. Let $H' = H - \{u_1, u_2, u_{11}, u_{12}, u_{21}, u_{22}\}$. By the minimality of H, H' has a strong edge-coloring with at most nineteen colors. One can observe that $L'(uu_1)| \geq 11$, $L'(uu_2)| \geq 11$, $L'(u_1u_{11})| \geq 14$, $L'(u_1u_{12})| \geq 14$, $L'(u_2u_{21})| \geq 14$, $L'(u_2u_{22})| \geq 14$, $L'(u_{11}u_{21})| \geq 13$, $L'(u_{11}u_{22})| \geq 13$, $L'(u_{12}u_{21})| \geq 13$, $L'(u_{12}u_{22})| \geq 13$, $L'(u_{11}u'_{11})| \geq 7$, $L'(u_{12}u'_{12})| \geq 7$,$L'(u_{21}u'_{21})| \geq 7$, and $L'(u_{22}u'_{22})| \geq 7$. Thus, we can properly color $u_{11}u'_{11}, u_{12}u'_{12}, u_{21}u'_{21}, u_{22}u'_{22}, uu_1, uu_2, u_{11}u_{21}$,$u_{11}u_{22}, u_{12}u_{21}, u_{12}u_{22}$, u_1u_{11}, $u_1u_{12}, u_2u_{21}, u_2u_{22}$ in turn and obtain a strong edge-coloring with nineteen colors, a contradiction. This proves Claim 2.

By Claims 1 and 2, we assume that the distance between u_1u_{11} and u_2u_{21} is at least 3. In order to prove Lemma 2.8, we need the following claim.

Claim 3. One of the following holds.

(1) There is a pair of non adjacent vertices u_{12} and u_{2j} for some $j \in \{1, 2\}$.

(2) There is a pair of non adjacent vertices u_{1i} and u_{21} for some $i \in \{1, 2\}$.

Proof of Claim 3. By symmetry, we only prove (1). Suppose otherwise that for each $j \in \{1, 2\}$, $u_{12}u_{2j} \in E(G)$. Let $N(u_{12}) = \{u_1, u_{21}, u_{22}, u'_{12}\}$, $N(u_{21}) = \{u_2, u_{12}, u'_{21}, u''_{21}\}$ and $N(u_{22}) = \{u_2, u_{12}, u'_{22}, u''_{22}\}$. Let $H' = H - \{u_1, u_2, u_{12}, u_{21}, u_{22}\}$. By the minimality of H, H' has a strong edge-coloring with at most nineteen colors. One can observe that $|L'(uu_1)| \geq 8$, $|L'(uu_2)| \geq 11$, $|L'(u_1u_{11})| \geq 7$, $|L'(u_1u_{12})| \geq 13$, $|L'(u_{12}u_{21})| \geq 10$, $|L'(u_{12}u_{22})| \geq 10$, $|L'(u_2u_{21})| \geq 13$,

$|L'(u_2u_{22})| \geq 13$, $|L'(u_{12}u'_{12})| \geq 7$, $|L'(u_{21}u'_{21})| \geq 7$, $|L'(u_{21}u''_{21})| \geq 7$, $|L'(u_{22}u'_{22})| \geq 7$, and $|L'(u_{22}u''_{22})| \geq 7$. Thus, we can properly color $u_{12}u'_{12}, u_{21}u'_{21}, u_{21}u''_{21}, u_{22}u'_{22}, u_{22}u''_{22}, u_1u_{11}, uu_1, u_{12}u_{21}, u_{12}u_{22}, uu_2, u_2u_{21}, u_2u_{22}$, u_1u_{12} in turn and obtain a strong edge-coloring with nineteen colors, a contradiction. This proves Claim 3.

Let $H' = H - \{u, u_1, u_2\}$. By the minimality of H, H' has a strong edge-coloring with at most nineteen colors. One can observe that $|L'(uu_3)| \geq 4, |L'(uu_4)| \geq 4, |L'(uu_1)| \geq 7, |L'(uu_2)| \geq 7, |L'(u_1u_{11})| \geq 4, |L'(u_1u_{12})| \geq 4, |L'(u_2u_{21})| \geq 4$ and $|L'(u_2u_{22})| \geq 4$.

Claim 4. (1) $L'(u_1u_{11}) \cap L'(u_2u_{21}) = \emptyset$.

(2) Either $L'(u_1u_{12}) \cap L'(u_2u_{2j}) = \emptyset$ for some $j \in \{1, 2\}$ or $L'(u_2u_{21}) \cap L'(u_1u_{1i}) = \emptyset$ for some $i \in \{1, 2\}$.

Proof of Claim 4. We only prove (1) and the proof of (2) is similar. Suppose otherwise that $\alpha \in L'(u_1u_{11}) \cap L'(u_2u_{21})$ by symmetry. We assign α to both u_1u_{11} and u_2u_{21}, then properly color u_1u_{12}, uu_3, uu_4. By Claim 1, $u_{12} \neq u_{22}$. If u_{12} is adjacent to u_{22}, then one can observe that $|L'(uu_3)| \geq 4, |L'(uu_4)| \geq 4, |L'(uu_1)| \geq 7, |L'(uu_2)| \geq 7, |L'(u_1u_{11})| \geq 4, |L'(u_1u_{12})| \geq 5, |L'(u_2u_{21})| \geq 4$ and $|L'(u_2u_{22})| \geq 5$. So, we can properly color u_2u_{22}. Thus, we may assume that the distance between u_1u_{12} and u_2u_{22} is at least 3. In this case, we also properly color u_2u_{22}. In each case, since $|L'(uu_1)| \geq 7$ and $|L'(uu_2)| \geq 7$, we can properly color uu_1, uu_2. Thus, we obtain a strong edge-coloring with nineteen colors, a contradiction. This proves our claim.

By Claim 4, we may assume that $|L'(u_1u_{11}) \cup L'(u_2u_{21})| \geq 8$ and that either $|L'(u_1u_{12} \cup L'(u_2u_{2j})| \geq 8$ for some $j \in \{1, 2\}$ or $|L'(u_1u_{1i}) \cup L'(u_2u_{21})| \geq 8$ for some $i \in \{1, 2\}$. For any subset $T \subseteq \{uu_1, uu_2, uu_3, uu_4, u_1u_{11}, u_1u_{12}, u_2u_{21}, u_2u_{22}\}$, $|\sum_{e \in T} L'(e)| \geq |T|$. By Theorem 1.5, we can assign eight distinct colors to eight uncolored edges to obtain a strong-edge coloring with nineteen colors, a contradiction. ■

Consider the final weight of x_0. By Lemma 2.2, x_0 is a 3-vertex. By Lemma 2.1 (c), x_0 is adjacent to at least three 4-vertices, and by Lemmas 2.1 (d) and 2.8, none of which is a 4_3-vertex or a 4_4-vertex or 4_2-vertex. Furthermore, x_0 is adjacent to at most one 4_1-vertex.

Since x_0 is not adjacent to a 4_2-vertex, $\omega^*(v) = 3 - \frac{15}{4} + 3(4 - \frac{15}{4}) = 15 - 4 \cdot \frac{15}{4} = 0$, a contradiction.

2.5 Case 5: $(m, k) = (\frac{51}{13}, 20)$

Lemma 2.9 *The distance between two 3-vertices is at least 4.*

Proof. By Lemma 2.1 (d), the distance between two 3-vertices is at least 3. Suppose that there exists the distance between two 3-vertices v and y at distance 3. Let $N(v) = \{u, w, t\}$, $N(t) = \{v, t_1, t_2, x\}$, and $N(x) = \{t, x_1, x_2, y\}$ (see Figure 8).

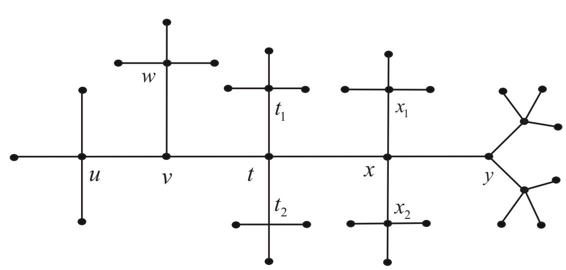

Figure 8: The distance between two 3-vertices v and y is 3

By Lemma 2.1 (d), $wy \notin E(H)$. By the minimality of H, $H' = H - v$ has a strong edge-coloring c with at most twenty colors. In the strong edge-coloring c of H', we erased the colors of edges tx and xy so that we get a partial coloring c'. Observe that $|L'(uv)| \geq 3$, $|L'(vw)| \geq 3$, $|L'(vt)| \geq 4$, $|L'(tx)| \geq 2$ and $|L'(xy)| \geq 2$. If $L'(xy) \cap L'(vw) \neq \emptyset$, we color edges xy, vw with the same color and then color tx, uv, vt in turn and we obtain a desired strong edge-coloring with twenty colors, a contradiction. If $L'(xy) \cap L'(vw) = \emptyset$, then $|L'(xy) \cup L'(vw)| \geq 5$. Let $T = \{uv, vw, vt, tx, xy\}$, for any $S \subseteq T$, we have $| \cup_{e \in S} L'(e)| \geq |S|$. By Theorem 1.5, we can assign a distinct color to uncolored edge, then we obtain a desired strong edge-coloring with twenty colors, a contradiction. ■

Lemma 2.10 *The distance between two 3-vertices is at least 5.*

Proof. By Lemma 2.9, the distance between two 3-vertices is at least 4. Suppose otherwise that there exist two 3-vertices v and x at distance 4. We use the notations in Figure 9.

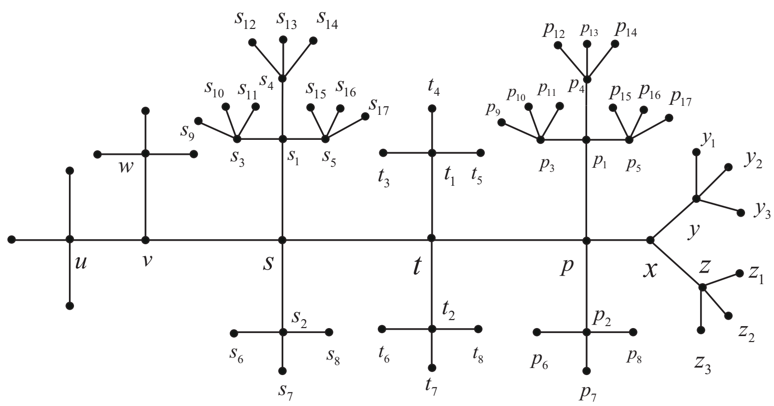

Figure 9: The distance between two 3-vertices v and x is 4

By the minimality of H, $H' = H - \{v, x\}$ has a strong edge-coloring c with at most twenty colors. In the strong edge-coloring c of H', we erased the colors of edges st and tp so that we get a partial coloring c'. We will extend this partial coloring c' to a strong edge-coloring of H. One can observe that $|L'(uv)| \geq 3$, $|L'(vw)| \geq 3$, $|L'(vs)| \geq 4$, $|L'(st)| \geq 2$, $|L'(tp)| \geq 2$, $|L'(px)| \geq 4$, $|L'(xy)| \geq 3$, $|L'(xz)| \geq 3$. We consider the following two cases.

Case 1. One of $L'(uv) \cap L'(tp)$, $L'(wv) \cap L'(tp)$, $L'(xy) \cap L'(st)$ and $L'(xz) \cap L'(st)$ is not empty.

We assume, without loss of generality, that $L'(uv) \cap L'(tp) \neq \emptyset$. We establish the following claim.

Claim 1. (1) $L'(uv) \cap L'(tp) \subseteq L'(px)$, $L'(uv) \cap L'(tp) \subseteq L'(xy)$ and $L'(uv) \cap L'(tp) \subseteq L'(xz)$.
(2) $L'(xy) \subseteq L'(px)$ and $L'(xz) \subseteq L'(px)$.
(3) $|L'(px)| = 4$, $|L'(xy)| = 3$ and $|L'(xz)| = 3$.
(4) $L'(xy) = L'(xz)$.
(5) $L'(st) \subseteq L'(px)$ and $|L'(st)| = 2$.
(6) $|L'(st) \cap L'(xy)| = 1$.
(7) $|L'(vs)| = 4$, $|L'(uv)| = 3$ and $|L'(vw)| = 3$.

Proof of Claim 1. (1) We only prove that $L'(uv) \cap L'(tp) \subseteq L'(px)$. The proofs for other cases are similar. Suppose otherwise we can pick $\alpha \in L'(uv) \cap L'(tp)$ and $\alpha \notin L'(px)$, then we can color uv and tp with α and color st, wv, vs, xy, xz, px in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(2) We only prove that $L'(xy) \subseteq L'(px)$ and the proof for the other case is similar. Suppose otherwise. We can pick $\beta \in L'(xy)$ and $\beta \notin L'(px)$. By (1), $L'(uv) \cap L'(tp) \subseteq L'(px)$ and hence $\beta \notin L'(uv) \cap L'(tp)$. Since $L'(uv) \cap L'(tp) \neq \emptyset$, we color uv and tp with the same color, color xy with the color β and color st, wv, vs, xz, px in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(3) We only prove that $|L'(px)| = 4$ and the proofs for other cases are similar. Suppose otherwise that $|L'(px)| \geq 5$. We can color uv and tp with the same color and color st, wv, vs, xz, xy, px in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(4) If $L'(xy) \neq L'(xz)$, then we have $L'(xy) \cup L'(xz) = L'(px)$ since $L'(xy) \subseteq L'(px)$ and $L'(xz) \subseteq L'(px)$. Thus we can color uv and tp with the same color and color st, wv, vs in turn so that we get a partial coloring c''. One can observe that $|L'(px) \setminus \{c''(tp), c''(st\}| \geq 2$, $|L'(xy) \setminus \{c''(tp)\}| = 2$, $|L'(xz) \setminus \{c''(tp)\}| = 2$ and $|L'(xy) \cup L'(xz) \setminus \{c''(tp)\}| = 3$. By Theorem 1.5, we can assign distinct colors to xy, xz and px. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(5) Suppose otherwise we can pick $\beta' \in L'(st)$ and $\beta' \notin L'(px)$. By (1), $L'(uv) \cap L'(tp) \subseteq L'(px)$. Then $\beta' \notin L'(uv) \cap L'(tp)$. Thus, we color uv and tp with the same color, color st with the color β' and color wv, vs, xy, xz, px in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. Suppose otherwise that $|L'(st)| \geq 3$. We color uv and tp with the same color, then color xy, xz, xp, st, vw and vs in turn. Therefore, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(6) We first show that $L'(st) \cap L'(xy) \neq \emptyset$. Suppose otherwise that $L'(st) \cap L'(xy) = \emptyset$. By (2) and (5), $L'(xy) \subseteq L'(px)$ and $L'(st) \subseteq L'(px)$. This implies that $|L'(px)| \geq |L'(xy)| + |L'(st)| \geq 5$, contrary to (3). We now show that $|L'(st) \cap L'(xy)| = 1$. Suppose otherwise that $|L'(st) \cap L'(xy)| \geq 2$. We color uv and tp with the same color α^* and we can pick $\beta'' \in L'(st) \cap L'(xy) \setminus \{\alpha^*\}$. Thus we color st, xy with the same color β'' and color wv, vs, xz and px in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(7) By (6), $L'(st) \cap L'(xy) \neq \emptyset$. By replacing that $L'(uv) \cap L'(tp) \neq \emptyset$ by that $L'(st) \cap L'(xy) \neq \emptyset$, we obtain $|L'(vs)| = 4$, $|L'(uv)| = 3$ and $|L'(vw)| = 3$ by the argument in the proof of (3).

So far, we have proved Claim 1.

By Claim 1(4), we assume, without loss of generality, that $L'(px) = \{1, 2, 3, 4\}$, $L'(xy) = L'(xz) = \{1, 2, 3\}$. By Claim 1(5), we assume, without loss of generality, that $L'(st) = \{3, 4\}$. By Claim 1(7), we may assume, without loss of generality, that $L'(vs) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $L'(uv) = L'(wv) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $L'(tp) = \{\alpha_3, \alpha_4\}$.

We claim that $L'(tp) = \{\alpha_3, \alpha_4\} = \{3, 4\}$. If $3 \notin L'(tp)$, then $3 \in L(tp)$. Since $L'(st) = \{3, 4\}$ and $L'(px) = \{1, 2, 3, 4\}$, $3 \notin L(st) \cup L(px)$ and $L(tp) \subseteq L(st) \cup L(px)$. This implies that $3 \notin L(tp)$, a contradiction. Thus, $3 \in L'(tp)$. By symmetry, we may assume that $4 \in L'(tp)$. If $L'(vs) = \{\alpha_1, \alpha_2, 3, 4\}$ and $L'(uv) = L'(wv) = \{\alpha_1, \alpha_2, 4\}$, then we color both tp and uv with 4, color both st and xy with 3 and color wv with α_1, color vs with α_2, color xz with 1 and color px with 2. This means that we obtain a desired strong edge-coloring with twenty colors, a contradiction. Therefore, we may assume that $L'(vs) = \{\alpha_1, \alpha_2, 3, 4\}$, $L'(uv) = L'(wv) = \{\alpha_1, \alpha_2, 3\}$.

Recall that $L'(px) = \{1, 2, 3, 4\}$. We may assume, without loss of generality, that $c'(pp_1) = 5$, $c'(pp_2) = 6$, $c'(p_1p_3) = 7$, $c'(p_1p_4) = 8$, $c'(p_1p_5) = 9$, $c'(p_2p_6) = 10$, $c'(p_2p_7) = 11$, $c'(p_2p_8) = 12$, $c'(tt_1) = 13$, $c'(tt_2) = 14$, $c'(yy_1) = 15$, $c'(yy_2) = 16$, $c'(yy_3) = 17$, $c'(zz_1) = 18$, $c'(zz_2) = 19$, $c'(zz_3) = 20$. We now claim that $\{15, 16, 17, 18, 19, 20\} \subseteq \{p_3p_9, p_3p_{10}, p_3p_{11}, p_4p_{12}, p_4p_{13}, p_4p_{14}, p_5p_{15},$

$p_5p_{16}, p_5p_{17}\}$. If $15 \notin \{p_3p_9, p_3p_{10}, p_3p_{11}, p_4p_{12}, p_4p_{13}, p_4p_{14}, p_5p_{15}, p_5p_{16}, p_5p_{17}\}$, we recolor pp_1 with color 15, color st with 5, color pt with 3, color xy with 1, color xz with 2, color px with 4, color vs with 4, color uv with 3 and color wv with a color $\alpha^{**} \in \{\alpha_1, \alpha_2\}$ and $\alpha^{**} \neq 5$. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. Similarly, we can prove that $\{16, 17, 18, 19, 20\} \subseteq \{p_3p_9, p_3p_{10}, p_3p_{11}, p_4p_{12}, p_4p_{13}, p_4p_{14}, p_5p_{15}, p_5p_{16}, p_5p_{17}\}$.

By symmetry, we may assume that $c'(p_3p_9) = 15$, $c'(p_3p_{10}) = 16$, $c'(p_3p_{11}) = 17$, $c'(p_4p_{12}) = 18$, $c'(p_4p_{13}) = 19$, $c'(p_4p_{14}) = 20$. Now we claim that $5 \in \{\alpha_1, \alpha_2\}$. If $5 \notin \{\alpha_1, \alpha_2\}$, we can pick $\beta^* \in \{1, 2, 3, 4\} \backslash \{c'(p_5p_{15}), c'(p_5p_{16}), c'(p_5p_{17})\}$. If $\beta^* \in \{1, 2\}$, then we recolor pp_1 with β^* and color both st and xy with 5, color pt with 4, color xz with the color in $\{1, 2\} \setminus \{\beta^*\}$, color both px and vs with 3, color uv and wv with α_1 and α_2 respectively. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. If $\beta^* \in \{3, 4\}$, then we recolor pp_1 with β^* and color both st and xy with 5, color pt with a color in $\{3, 4\} \setminus \{\beta^*\}$, color xz with 1, color px with 2, color vs with α_1, color uv and wv with 3 and α_2, respectively. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

By symmetry, we have $6 \in \{\alpha_1, \alpha_2\}$. Therefore we have $L'(vs) = \{3, 4, 5, 6\}$, $L'(uv) = L'(wv) = \{3, 5, 6\}$. Since $L'(px) = \{1, 2, 3, 4\}$, $L'(xy) = L'(xz) = \{1, 2, 3\}$ and by symmetry, we claim that $\{c'(ss_1), c'(ss_2)\} = \{1, 2\}$.

Now we claim that $\{c'(p_5p_{15}), c'(p_5p_{16}), c'(p_5p_{17})\} = \{1, 2, 4\}$. If $1 \notin \{c'(p_5p_{15}), c'(p_5p_{16}), c'(p_5p_{17})\}$, we recolor pp_1 with 1 and color both st and xy with 5, color pt with 3, color xz with 2, color both px and vs with 4, color uv with 3 and color wv with 6. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. Similarly, we can prove that $2 \in \{c'(p_5p_{15}), c'(p_5p_{16}), c'(p_5p_{17})\}$.

If $4 \notin \{c'(p_5p_{15}), c'(p_5p_{16}), c'(p_5p_{17})\}$, we recolor pp_1 with 4 and color both st and xy with 5, color pt with 3, color xz with 1, color px with 2 and vs with 4, color uv with 3 and color wv with 6. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

Recall $L'(st) = L'(tp) = \{3, 4\}$ and $\{c'(ss_1), c'(ss_2)\} = \{1, 2\}$. We assume, without loss of generality, that $c'(t_1t_3) = 15$, $c'(t_1t_4) = 16$, $c'(t_1t_5) = 17$, $c'(t_2t_6) = 18$, $c'(t_2t_7) = 19$, $c'(t_2t_8) = 20$. $c'(s_1s_3) = 7$, $c'(s_1s_4) = 8$, $c'(s_1s_5) = 9$, $c'(s_2s_6) = 10$, $c'(s_2s_7) = 11$, $c'(s_2s_8) = 12$. By symmetry of p and s, we may assume that $\{c'(s_3s_9), c'(s_3s_{10}), c'(s_3s_{11}), c'(s_4s_{12}), c'(s_4s_{13}), c'(s_4s_{14}), c'(s_5s_{15}), c'(s_5s_{16}), c'(s_5s_{17})\} = \{15, 16, 17, 18, 19, 20, 4, 5, 6\}$. Therefore, we can recolor both ss_1 and pp_1 with 3, color st with 1, color tp with 5, color xy, xz, px, uv, wv, vs with 1, 2, 4, 5, 6, 4, respectively, Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

Case 2. $L'(uv) \cap L'(tp) = \emptyset$, $L'(wv) \cap L'(tp) = \emptyset$, $L'(xy) \cap L'(st) = \emptyset$ and $L'(xz) \cap L'(st) = \emptyset$.

In this case, we have $|L'(uv) \cup L'(tp)| \geq 5$, $|L'(wv) \cup L'(tp)| \geq 5$, $|L'(xy) \cup L'(st)| \geq 5$, $|L'(xz) \cup L'(st)| \geq 5$. We now prove the following claim.

Claim 2. (1) $|L'(vs)| = 4$ and $|L'(px)| = 4$.

(2) $|L'(uv)| = |L'(wv)| = 3$.

(3) $L'(uv) \subseteq L'(vs)$, $L'(wv) \subseteq L'(vs)$, $L'(xy) \subseteq L'(px)$ and $L'(xz) \subseteq L'(px)$.

(4) $L'(uv) = L'(wv)$.

Proof of Claim 2. (1) We only prove that $|L'(vs)| = 4$ and the proof for the case $|L'(px)| = 4$ is similar. Suppose otherwise that $|L'(vs)| \geq 5$. In this case, let $T = \{st, tp, px, xy, xz\}$. Note that $|L'(st) \cup L'(xy)| \geq 5$. For any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq S$. By Theorem 1.5, we can assign a distinct color to each edge in T. We then properly color uv, vw and vs in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(2) Suppose otherwise that $|L'(uv)| \geq 4$. The proofs for the cases are similar. In this case, we also let $T = \{st, tp, px, xy, xz\}$. Since $|L'(xy) \cup L'(st)| \geq 5$, for any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq S$.

By Theorem 1.5, we can assign a distinct color to each edge in T. We now properly color vw, vs and uv in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(3) Suppose otherwise that $L'(uv) \nsubseteq L'(vs)$. The proofs for the other cases are similar. Let $\gamma \in L'(uv) \setminus L'(vs)$. Let $T = \{st, tp, px, xy, xz\}$. Since $|L'(xy) \cup L'(st)| \geq 5$, for any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq S$. By Theorem 1.5, we can assign a distinct color to each edge in T. In particular, st is assigned color β. If $\gamma \neq \beta$, then we now color vw with a color in $L'(vw) \setminus \{\gamma, \beta\}$, properly color vw and vs in turn; if $\gamma = \beta$, then properly color uv, vw and vs in turn. In both cases, Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction.

(4) Suppose otherwise that $L'(uv) \neq L'(wv)$. Let $\alpha \in L'(uv) \setminus L'(vw)$ and $\beta \in L'(vw) \setminus L'(uv)$. Let $T = \{st, tp, px, xy, xz\}$. Since $|L'(xy) \cup L'(st)| \geq 5$, for any $S \subseteq T$, we have $|\cup_{e \in S} L'(e)| \geq S$. By Theorem 1.5, we can assign a distinct color to each edge in T. In particular, st and tp are assigned color γ_1 and γ_2, respectively. Since $\alpha \neq \beta$, we may assume that $\alpha \neq \gamma$. Now we color vs with a color in $L'(vs) \setminus \{\gamma_1, \gamma_2, \alpha\}$, properly color vw and color uv with color α. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. By symmetry, $L'(xy) = L'(xz)$.

We now complete the proof of Claim 2.

By Claim 2, we assume, without loss of generality, that $L'(uv) = L'(wv) = \{1, 2, 3\}$, $L'(vs) = \{1, 2, 3, 4\}$, $L'(xy) = L'(xz) = \{\beta_1, \beta_2, \beta_3\}$, $L'(px) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$.

Since $L'(uv) \cap L'(pt) = \emptyset$, $|L'(pt)| \geq 2$, we can pick $\gamma^{**} \in L'(pt)$ and $\gamma^{**} \neq 4$. If $\gamma^{**} = \beta_1$, we firstly color tp with γ^{**}, color xy, xz,and px with β_2, β_3 and β_4 respectively. Since $L'(xy) \cap L'(st) = \emptyset$, $\{\beta_1, \beta_2, \beta_3\} \cap L'(st) = \emptyset$. This implies that $\gamma^{**} \notin L'(st)$. Thus, we can properly color st, uv, wv, vs in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. The proofs are similar for the cases that $\gamma^{**} = \beta_2$ and $\gamma^{**} = \beta_3$. If $\gamma^{**} = \beta_4$, we firstly color tp with γ^{**}, color xy, xz,and px with β_1, β_2 and β_3 respectively. Since $L'(xy) \cap L'(st) = \emptyset$, we can properly color st, uv, wv, vs in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. If $\gamma^{**} \notin \{\beta_1, \beta_2, \beta_3, \beta_4\}$, then we firstly color tp with γ^{**}. Since $\gamma^{**} \neq 4$ and $L'(st) \cap L'(xy) = \emptyset$, we can properly color st, xy, xz, px in turn. Since $L'(uv) \cap L'(pt) = \emptyset$, $L'(wv) \cap L'(pt) = \emptyset$ and $\gamma^{**} \in L'(pt)$, we can color uv, wv, vs in turn. Thus, we obtain a desired strong edge-coloring with twenty colors, a contradiction. ■

Consider the final weight of x_0. By Lemma 2.2, x_0 is a 3-vertex. By Lemma 2.10, for each 4-vertex $u \in N_2(x_0)$, x_0 is the only one 3^--vertex in $N_2(u)$. By (R2), $\omega^*(x_0) = 3 - \frac{51}{13} + 3 \cdot (4 - \frac{51}{13}) + 9 \cdot (4 - \frac{51}{13}) = 51 - 13 \cdot \frac{51}{13} = 0$, a contradiction.

3 Concluding remarks

We feel that Lemma 2.10 can be strengthened to show that the distance between 3-vertices should be arbitrary large, implying that there is at most one 3-vertex. But one may have an argument to show there is no 3-vertex at all, so we do not make much more effort than Lemma 2.10.

We do not have constructions to show the sharpness of the maximum average degrees in our theorem, and we do not believe they are sharp.

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