PLANAR GRAPHS WITHOUT CYCLES OF LENGTH 4 OR 5 ARE $(3,0,0)$-COLORABLE

OWEN HILL*, DIANA SMITH†, YINGQIAN WANG‡, LINGJI XU‡, AND GEXIN YU*

Abstract. We study Steinberg's Conjecture. A graph is $(c_1, c_2, \cdots, c_k)$-*colorable* if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that for every i with $1 \le i \le k$ the subgraph $G[V_i]$ has maximum degree at most c_i. We show that every planar graph without 4- or 5-cycles is $(3,0,0)$-colorable. This is a relaxation of Steinberg's Conjecture that every planar graph without 4- or 5-cycles is properly 3-colorable (i.e., $(0,0,0)$-colorable).

1. Introduction

Graph Colorings have been studied extensively over the past century. Most famously, Appel and Haken [1, 2] proved that every planar graph is properly 4-colorable in 1977. However, the problem of deciding whether a planar graph is properly 3-colorable is NP-complete [8]. In 1959, Grötzsch [9] proved the well-known theorem that planar graphs without 3-cycles are properly 3-colorable. A lot of research was devoted to find sufficient conditions for a planar graph to be 3-colorable, by allowing a triangle together with some other conditions, for example. One of such efforts is the following famous conjecture made by Steinberg in 1976.

Conjecture 1 (Steinberg, [12])**.** *All planar graphs without* 4*-cycles and* 5*-cycles are properly* 3*-colorable.*

Not much progress in this direction was made until Erdős proposed to find a constant C such that a planar graph without cycles of length from 4 to C is properly 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [4] showed that $C \le 7$. For more results, see the recent nice survey by Borodin [3].

Yet another direction of relaxation of the conjecture is to allow some defects in the color classes. A graph is $(c_1, c_2, \cdots, c_k)$-*colorable* if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that for every $i : 1 \le i \le k$ the subgraph $G[V_i]$ has maximum degree at most c_i. Thus a $(0,0,0)$-colorable graph is properly 3-colorable.

Cowen, Cowen, and Woodall [6] proved that planar graphs are $(2,2,2)$-colorable. Eaton and Hull [7] and independently Škrekovski [11] showed that every planar graph is $(2,2,2)$-choosable. Xu [13] proved that all planar graphs without adjacent triangles or 5-cycles are $(1,1,1)$-colorable. Chang, Havet, Montassier, and Raspaud [5] proved that all planar graphs without 4-cycles or 5-cycles are $(2,1,0)$-colorable and $(4,0,0)$-colorable. Xu and Wang [15]

Date: June 2, 2013.

showed that planar graphs without 4- or 6-cycles are $(3,0,0)$- and $(1,1,0)$-colorable. Hill and Yu [10], and independently Xu, Miao, and Wang [14] improved one of the results by Chang et. al. and showed that all planar graphs without 4-cycles or 5-cycles are $(1,1,0)$-colorable. In this paper, we prove the following relaxation of the Steinberg Conjecture and improve the other result of Chang et al.

Theorem 1. *All planar graphs without* 4*-cycles or* 5*-cycles are* $(3,0,0)$*-colorable.*

We will use the following notations in the proofs. A *k-vertex* (*k^+-vertex*, k^--vertex) is a vertex of degree k (at least k, at most k resp.). The same notation will apply to faces. An *$(\ell_1, \ell_2, \ldots, \ell_k)$-face* is a k-face with incident vertices of degree $\ell_1, \ell_2, \ldots, \ell_k$. A *bad* 3*-vertex* is a 3-vertex on a 3-face. A face f is a *pendant* 3*-face* to vertex v if v is not on f but is adjacent to some bad 3-vertex on f. The *pendant neighbor* of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face.

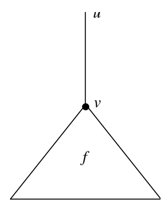

FIGURE 1. In the figure, v is a bad 3-vertex, f is a pendant 3-face to u, and u is the pendant neighbor of v.

A vertex v is *properly colored* if all neighbors of v have different colors from v. A vertex v is *nicely colored* if it shares a color with at most $\max\{s_i - 1, 0\}$ neighbors, where s_i is the deficiency allowed for color i, thus if a vertex v is nicely colored by a color c which allows deficiency $s_i > 0$, then an uncolored neighbor of v can be colored by c.

In the next section, we will prove some necessary reducible configurations, and in the last section, we finish the proof by using a discharging argument.

2. Reducible Configurations

Let G be a minimum counterexample to Theorem 1, that is, G is a planar graph without 4- or 5-cycles and is not $(3,0,0)$-colorable, but any proper subgraph of G is $(3,0,0)$-colorable. We may assume that vertices colored by 1 may have up to three neighbors colored by 1.

The following are some simple observations about the minimal counterexamples to the above theorem.

Proposition 1. *(a) G contains no 2^--vertices.*
(b) a k-vertex in G can have $\alpha \le \lfloor \frac{k}{2} \rfloor$ incident 3*-faces, and at most $k - 2\alpha$ pendant* 3*-faces.*

The following is a very useful tool to extend a coloring on a subgraph of G to include more vertices.

Lemma 1. *Let H be a proper subgraph of G. Given a $(3,0,0)$-coloring of $G-H$, if exactly two neighbors of $v \in H$ are colored so that one is a 5^{-}-vertex and the other is nicely colored, then there exists a $(3,0,0)$-coloring of $G-H$ that can be extended to $G-(H-v)$ such that v is nicely colored by 1.*

Proof. Let H be a subgraph of G such that $G-H$ has a $(3,0,0)$-coloring. Let $v \in H$ have neighbors u and w that are colored. Let $d(u) \leq 5$ and w be nicely colored. Color v by 1. Since w is nicely colored, if this coloring is invalid, then u must be colored by 1. In addition, u must have at least 3 neighbors colored by 1. To avoid recoloring u by 2 or 3, u must have at least one neighbor of color 2 and at least one neighbor of color 3. This implies that $d(u) \geq 6 > 5$, a contradiction. So v is colorable by 1. In addition, since the deficiency of color 1 is 3 and v only has 2 colored neighbors, v is nicely colored. ☐

Lemma 2. *Every 3-vertex in G has a 6^{+}-vertex as a neighbor.*

Proof. Let v be a 3-vertex in G such that each neighbor of v has degree at most 5. By the minimality of G, $G-v$ is $(3,0,0)$-colorable. If two vertices in the neighborhood of v share the same color, then v can be properly colored, so we can assume that all the neighbors of v are colored differently. Let u be the neighbor of v that is colored by 1. Then u must have 3 neighbors colored by 1 to forbid v to be colored by 1. In addition, u must have neighbors colored by 2 and 3 to forbid recoloring u by 2 or 3 and then coloring v by 1. Then, u has at least 6 neighbors, a contradiction. ☐

Call a $(3,3,3^{+})$-face *poor* if the pendant neighbors of the two 3-vertices have degrees at most 5. A $(3,3^{+},3^{+})$-face is *semi-poor* if exactly one of the pendant neighbors of the 3-vertices has degree 5 or less. A 3-face is *non-poor* if each 3-vertex on it, if any, has the pendant neighbor being a 6^{+}-vertex. Finally, a *poor 3-vertex* is a 3-vertex on a poor or semi-poor 3-face that has a 5^{-}-vertex as its pendant neighbor.

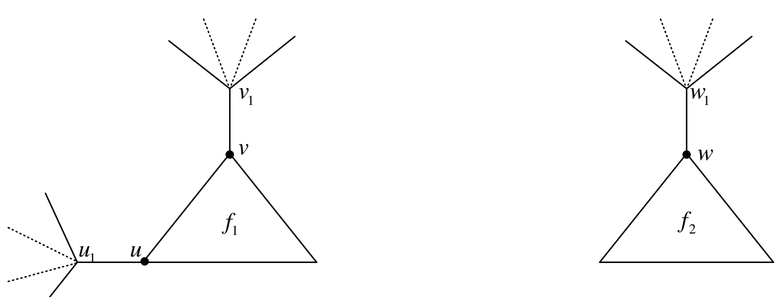

FIGURE 2. In the figure, f_1 is a poor 3-face and f_2 is a semi-poor 3-face.

Lemma 3. *All (3,3,6^{-})-faces in G are non-poor.*

Proof. For all $(3,3,5^-)$-faces in G, the proof is trivial by Lemma 2. Let uvw be a $(3,3,6)$-face in G with $d(u) = d(v) = 3$ such that the pendant neighbor v' of v has degree at most 5. By the minimality of G, $G\backslash\{u,v\}$ is $(3,0,0)$-colorable. Properly color u and color v differently than both w and v'. Then either we obtain a $(3,0,0)$-coloring of G, contradicting the choice of G, or u and v are both colored by 2 or 3, w.l.o.g. assume 2. This means that u' and v' share the same color (where u' is the pendant neighbor of u), different from the color of w.

Let w be colored by 1, then to avoid being able to recolor u or v by 1, w must have 3 outer neighbors colored by 1. Then w can be recolored by 2 or 3 depending on the color of its fourth colored neighbor. We recolor w by 2 or 3 and recolor u and v by 1 to get a coloring of G, a contradiction.

So we may assume that w is colored by 3, and that u' and v' are colored by 1. To avoid recoloring v by 1, v' must have at least 3 neighbors colored by 1. In addition, to avoid recoloring v' by 2 or 3 and coloring v by 1, v' must have neighbors colored by both 2 and 3. This contradicts that v' has degree less than 6. □

Here is a simple fact on extending a coloring to a poor 3-face.

Lemma 4. *Let $f = uvw$ be a poor 3-face with $d(u) = d(v) = 3$. Then a partial coloring of $G - \{u,v,w\}$ can be extended to include u and v so that u and v are colored with 1.*

Proof. Let u' and v' be the pendant neighbors of u and v, respectively. We may assume that u' and v' are colored, and as $d(u'), d(v') \leq 5$, we may further assume that u' and v' are both nicely colored (if not, then color 2 or 3 would be available to recolor them). So we can first color u by 1, and then by Lemma 1, color v by 1 as well. □

Lemma 5. *No 4^+-vertex $v \in V(G)$ can have $\lfloor \frac{d(v)}{2} \rfloor$ incident poor 3-faces.*

Proof. Let v be a k-vertex in G with $\lfloor \frac{k}{2} \rfloor$ incident poor $(3,3,k)$-faces. Let $u_1, u_2, \cdots, u_k$ be the neighbors of v, and let u_i' be the pendant neighbor if u_i is in a poor 3-face. Note that $d(u_i') \leq 5$ and we know that all except possibly u_k are in poor 3-faces.

By the minimality of G, $G\backslash\{v, u_1, u_2, \cdots, u_{k-1}\}$ is $(3,0,0)$-colorable. If $d(v)$ is odd, then by Lemma 4, for all i with $1 \leq i \leq k-1$, we can color u_i by 1, then properly color v to get a coloring of G. So we assume that $d(v)$ is even. By Lemma 4, for all i with $1 \leq i \leq k-2$, we can color u_i by 1. Then if u_k is colored by 1 we can color u_{k-1} properly and v properly to get a coloring of G. If u_k is colored by 2 or 3, then it is colored properly and by Lemma 1 we can color u_{k-1} by 1. Then we can properly color v to get a coloring of G, a contradiction. □

Lemma 6. *If an 8-vertex v is incident with three poor $(3,3,8)$-faces, then it cannot be incident with a semi-poor face, nor two pendant 3-faces.*

Proof. Let v be an 8-vertex in G with 3 incident poor $(3,3,8)$-faces. Let $u_1, u_2, \cdots, u_6$ be the 3-vertices in the poor $(3,3,8)$-face and let $u_1', u_2', \cdots, u_6'$ be the corresponding pendant neighbors, respectively. We know that for all i with $1 \leq i \leq 6$, $d(u_i') \leq 5$.

(i) Let vu_7u_8 be the incident semi-poor face with u_7 being the poor 3-vertex. Then by the minimality of G, $G\backslash\{v, u_1, u_2, \cdots, u_7\}$ is $(3,0,0)$-colorable. By Lemma 4, $u_1, u_2, \cdots, u_6$ can be colored by 1. Then if u_8 is colored by 1, we can properly color u_7 and then v to get a coloring of G. So we may assume that u_8 is not colored by 1, in which case it is nicely colored and we may color u_7 with 1 by Lemma 1, and then properly color v to get a coloring of G, a contradiction.

(ii) Let u_7 and u_8 be the bad 3-vertices adjacent to v. Then $G\backslash\{v, u_1, u_2, \cdots, u_7, u_8\}$ is $(3,0,0)$-colorable, by the minimality of G. Properly color both u_7 and u_8. If either u_7 or u_8 is colored by 1 or both have the same color, then by Lemma 4, we may color $u_1, u_2, \cdots, u_6$ by 1 and then properly color v. So we may assume that u_7 is colored by 2 and u_8 is colored by 3. Then we properly color $u_1, u_2, \cdots, u_6$, and it follows that for each i with $1 \leq i \leq 3$, u_{2i-1} and u_{2i} must be colored differently. Then v can have at most 3 neighbors colored by 1, all properly colored, so v can be colored by 1, a contradiction. □

Lemma 7. *If a 7-vertex v is incident with two poor $(3,3,7)$-faces, then it cannot be (i) incident with a semi-poor $(3,6^-,7)$-face and adjacent to a pendant 3-face, or (ii) adjacent to three pendant 3-faces.*

Proof. Let v be a 7-vertex in G with 2 incident poor $(3,3,7)$-faces. Let u_1, u_2, u_3, and u_4 be the 3-vertices on the poor $(3,3,7)$-faces and let u'_1, u'_2, u'_3, and u'_4 be their corresponding pendant neighbors, respectively. We know that for all i with $1 \leq i \leq 4$, $d(u'_i) \leq 5$.

(i) Let vu_5u_6 be a semi-poor face with u_5 being a poor 3-vertex and $d(u_6) \leq 6$ and let u_7 be a bad 3-vertex adjacent to v. By the minimality of G, $G\backslash\{v, u_1, u_2, u_3, u_4, u_5, u_7\}$ is $(3,0,0)$-colorable. Since at this point u_6 has at most 4 colored neighbors, if u_6 is colored by 1 then either it is nicely colored or it can be recolored properly. If u_6 is not nicely colored, then recolor u_6 properly.

Color u_7 properly. If u_7 is colored by 1, then by Lemma 4, we can color $u_1, u_2, \cdots, u_5$ by 1 and then color v properly, a contradiction. So we may assume w.l.o.g. that u_7 is colored by 2. Color $u_1, u_2, \cdots, u_5$ properly. Then, for each i with $1 \leq i \leq 3$, u_{2i} and u_{2i-1} are colored differently and nicely. This leaves v with at most 3 neighbors colored by 1, all nicely, so we may color v by 1 to get a coloring of G, a contradiction.

(ii) Let u_5, u_6, and u_7 be the bad 3-vertices adjacent to v. By the minimality of G, $G\backslash\{v, u_1, \ldots, u_7\}$ is $(3,0,0)$-colorable. Properly color u_5, u_6, and u_7. If the set $\{u_5, u_6, u_7\}$ does not contain both colors 2 and 3, then by Lemma 4, we can color u_1, u_2, u_3, and u_4 by 1 and color v properly. So we can assume that both colors 2 and 3 appear on u_5, u_6, or u_7. This implies that at most one vertex is colored by 1. So we properly color u_1, u_2, u_3, and u_4. Then v has at most 3 neighbors colored by 1, all nicely, so we can color v by 1 to get a coloring of G, a contradiction. □

Lemma 8. *Let uvw be a semi-poor $(3,7,7)$-face in G such that $d(v) = d(w) = 7$. Then vertices v and w cannot both be 7-vertices that are incident with two poor 3-faces, one semi-poor $(3,7,7)$-face, and have one pendant 3-face.*

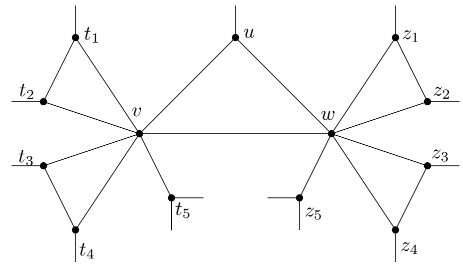

FIGURE 3. Figure for Lemma 8

Proof. Let uvw be a semi-poor $(3,7,7)$-face in G such that $d(v) = d(w) = 7$ and both v and w are incident with two poor 3-faces, one $(3,7,7)$-face, and adjacent to one pendant 3-face. Let the neighbors of v and w be $t_1, t_2, \cdots, t_5$ and $z_1, z_2, \cdots, z_5$, respectively such that t_5 and z_5 are bad 3-vertices (See figure 2).

By the minimality of G, $G \backslash \{u, v, w, t_1, t_2, \cdots, t_5, z_1, z_2, \cdots, z_5\}$ is $(3,0,0)$-colorable. By Lemma 4, we can color t_1, t_2, t_3, and t_4 by 1. Then properly color t_5, v, and z_5 in that order. Vertex v will not be colored by 1, so w.l.o.g. assume that v is properly colored by 2. If z_5 is colored by 1, then by Lemma 4 and Lemma 1, we can color z_1, z_2, z_3, z_4, and u by 1 and then properly color w, to get a coloring of G, a contradiction. So we can assume that z_5 is not colored by 1. Then we properly color z_1, z_2, z_3, z_4 and u, so w can have at most three neighbors colored by 1, all properly. We can color w by 1 to get a coloring of G, a contradiction. □

3. Discharging Procedure

We start the discharging process now. We let the initial charge of vertex $u \in G$ be $\mu(u) = 2d(u) - 6$, and the initial charge of face f be $\mu(f) = d(f) - 6$. Then by Euler's formula, we have

$$\sum_{v \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12. \tag{1}$$

Let a special semi-poor $(3,7,7^+)$-face (see Figure 4) is a semi-poor 3-face incident with a 7-vertex which is also incident with two poor 3-faces and adjacent to one pendant 3-face.

We introduce the following discharging rules:

(R1) Every 4-vertex gives 1 to each incident 3-face.
(R2) Every 5 or 6-vertex gives 2 to each incident 3-face.
(R3) every 6^+-vertex gives 1 to each adjacent pendant 3-face.

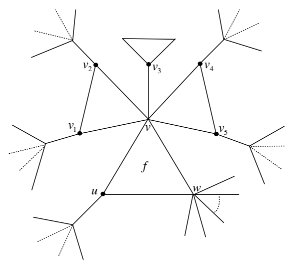

FIGURE 4. A special semi-poor $(3, 7, 7)$-face.

(R4) Each d-vertex with $7 \le d \le 10$ gives 3 to each incident poor $(3, 3, *)$-face, 2 to each incident semi-poor 3-face, except 7-vertices give 1 to each incident special semi-poor 3-face. Each d-vertex with $7 \le d \le 10$ gives 1 to all other incident 3-faces.

(R5) Every 11^+-vertex gives 3 to all incident 3-faces.

Now let v be a k-vertex. By Proposition 1, $k \ge 3$.

When $k = 3$, v is not involved in the discharging process, so $\mu^*(v) = \mu(v) = 0$.

When $k = 4$, by Proposition 1, v can have at most 2 incident 3-faces. By (R1), $\mu^*(v) \ge \mu(v) - 1 \cdot 2 = 0$.

When $k = 5$, by Proposition 1, v can have at most 2 incident 3-faces. By (R2), $\mu^*(v) \ge \mu(v) - 2 \cdot 2 = 0$.

When $k = 6$, by Proposition 1, v can have $\alpha \le 3$ incident 3-faces, and at most $(k - 2\alpha)$ pendant 3-faces. By (R2) and (R3), $\mu^*(v) \ge \mu(v) - 2 \cdot \alpha - 1 \cdot (k - 2\alpha) = k - 6 = 0$.

When $k = 7$, v has an initial charge $\mu(v) = 7 \cdot 2 - 6 = 8$. By Lemma 5, v has at most two poor 3-faces. If v has less than two incident poor 3-faces, then by (R3) and (R4), $\mu^*(v) \ge \mu(v) - 3 \cdot 1 - 1 \cdot 5 = 0$ since v gives at most one charge per vertex excluding vertices in poor 3-faces. So assume that v has exactly 2 incident poor 3-faces. By Lemma 7, v is adjacent to at most two pendant 3-faces, and if it is incident with a semi-poor $(3, 6^-, 7)$-face, then v is not adjacent to a pendant 3-face. So if v is not incident with a semi-poor $(3, 7^+, 7)$-face, then by (R3) and (R4), $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 2 \cdot 1 = 0$; if v is incident with a semi-poor $(3, 7^+, 7)$-face, then by rules (R3) and (R4), $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 1 \cdot 1 - 1 \cdot 1 = 0$.

When $k = 8$, v has an initial charge $\mu(v) = 8 \cdot 2 - 6 = 10$. By Lemma 5, v has at most three poor 3-faces. If v has less than 3 incident poor 3-faces, then by (R3) and (R4), $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 1 \cdot 4 = 10 - 6 - 4 = 0$ since v gives at most one charge per vertex excluding vertices in poor 3-faces. So let v be incident with exactly 3 poor 3-faces. By Lemma 6, v cannot be incident with a semi-poor 3-face or adjacent to two pendant 3-faces, then $\mu^*(v) \ge \mu(v) - 3 \cdot 3 - 1 \cdot 1 = 0$.

When $k = 9$, by Lemma 5, v is incident with at most three poor 3-faces. The worst case occurs when v is incident with three poor $(3, 3, 9)$-faces, one semi-poor $(3, 3, 9)$-face, and one pendant 3-face, or when v is incident with three poor $(3, 3, 9)$-faces and three pendant 3-faces. So by (R3) and (R4), $\mu^*(v) \ge \mu(v) - 1 \cdot 1 - 3 \cdot 3 - 2 \cdot 1 = 12 - 1 - 9 - 2 = 0$.

When $k = 10$, by Lemma 5, v is incident with at most four poor $(3,3,10)$-faces. So by (R3) and (R4), $\mu^*(v) \geq \mu(v) - 3 \cdot 4 - 2 \cdot 1 = 14 - 3 \cdot 4 - 2 \cdot 1 = 0$.

When $k \geq 11$, we assume that v is incident with α 3-faces, then by Proposition 1, $\alpha \leq \lfloor k/2 \rfloor$. Thus the final charge of v is $\mu^* \geq 2k - 6 - 3\alpha - 1 \cdot (k - 2\alpha) = k - \alpha - 6 \geq 0$.

Now let f be a k-face in G. By the conditions on G, $k = 3$ or $k \geq 6$. When $k \geq 6$, f is not involved in the discharging procedure, so $\mu^*(f) = \mu(f) = k - 6 \geq 0$. So in the following we only consider 3-faces. Recall that the minimum degree of G is at least three, so there is no $(2^-, 2^+, 2^+)$-faces.

Case 1: f is a $(4^+, 4^+, 4^+)$-face. By the rules, each 4^+-vertex on f gives at least 1 to f, so $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$.

Case 2: f is a $(3, 4^+, 4^+)$-face with vertices u, v, w such that $d(u) = 3$. If u is not a poor 3-vertex, then by (R3), f gains 1 from the pendant neighbor of u and by the other rules, f gains at least 2 from vertices on f, thus $\mu^*(f) \geq \mu(f) + 1 \cdot 3 = 0$. If u is a poor vertex (it follows that f is a semi-poor 3-face), then by Lemma 2, f is a $(3, 4^+, 6^+)$-face. Since v or w is a 6^+-vertex, it gives at least 2 to f unless f is a special semi-poor $(3, 7, 7^+)$-face, and as the other is a 4^+-vertex, it gives at least 1 to f. Therefore, if f is not a special semi-poor 3-face at v or w, then $\mu^*(f) \geq \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$; if f is a special semi-poor $(3, 7, 8^+)$-face, then f receives at least 2 from the 8^+-vertex, so $\mu^*(v) \geq \mu(v) + 2 \cdot 1 + 1 \cdot 1 = 0$. The only left case is that f is a special semi-poor $(3, 7, 7)$-face for both v and w (so that both v and w are incident with two poor 3-faces, one semi-poor $(3, 7, 7)$-face and adjacent to one pendant 3-face), but by Lemma 8, this situation is impossible.

Case 3: f is a $(3, 3, 4^+)$-face with 4^+-vertex v. If $d(v) \geq 11$, then by (R5), $\mu^*(f) \geq \mu(f) + 3 = 0$. So assume $d(v) \leq 10$. By Lemma 2, if $4 \leq d(v) \leq 5$, then each 3-vertex has the pendant neighbor of degree 6 or higher. And by Lemma 3, if $d(v) = 6$, then the face is non-poor. So by (R1) and (R3) (when $d(v) = 4$), $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$, or by (R1) and (R2) (when $d(v) > 4$), $\mu^*(f) \geq \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$.

Let $7 \leq d(v) \leq 10$. If f is poor, then by (R4), $\mu^*(f) = \mu(f) + 3 \cdot 1 = 0$. If f is semi-poor, then one 3-vertex on f is adjacent to a 6^+-vertex and thus by (R3) f gains 1 from it, together with the 2 that f gains from v by (R4), we have $\mu^*(f) = \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$. If f is non-poor, then both 3-vertices on f are adjacent to the pendant neighbors of degrees more than 5, thus by (R3) and (R4), $\mu^*(f) = \mu(f) + 1 \cdot 2 + 1 \cdot 1 = 0$.

Case 4: f is a $(3, 3, 3)$-face. By Lemma 2, each 3-vertex will have the pendant neighbor of degree 6 or higher, so by (R3), $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$.

Since for all $x \in V \cup F$, $\mu^*(x) \geq 0$, $\sum_{v \in V} \mu^*(v) + \sum_{f \in F} \mu^*(f) \geq 0$, a contradiction. This completes the proof of Theorem 1.2.

Acknowledgement

The research of the first and the last authors is supported in part by NSA grant H98230-12-1-0226 and NSF CSUMS grant. The research of the third and the fourth authors is supported in part by NSFC grant 11271335. The second author's research was done in association with the authors' Ph.D. dissertation, and the author would like to sincerely thank advisor Dr. N. Eaton for all her advice. The authors also would like to thanks the referees for their careful reading and valuable comments.

References

[1] K. Appel and W. Haken, Every planar map is four colorable. Part I. Discharging, *Illinois J. Math.*, **21** (1977), 429–490.

[2] K. Appel and W. Haken, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.*, **21** (1977) 491–567.

[3] O.V. Borodin, Colorings of planar graphs: a survey. *Disc. Math.*, **313** (2013) 517–539.

[4] O. V. Borodin, A. N. Glebov, A. R. Raspaud, and M. R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable. *J. of Comb. Theory, Ser. B*, **93** (2005), 303–311.

[5] G. Chang, F. Havet, M. Montassier, and A. Raspaud, Steinberg's Conjecture and near colorings, manuscript.

[6] L. J. Cowen, R. H. Cowen, and D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, *J. Graph Theory* **10** (1986), 187–195.

[7] N. Eaton and T. Hull, Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, **25** (1999), 78–87.

[8] Garey, Michael R. and Johnson, David S., (1979) Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, ISBN 0-7167-1045-5.

[9] H. Grötzsch, Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. (German) Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 1958/1959 109–120.

[10] O. Hill and G. Yu, A relaxation of Steiberg's Conjecture, http://arxiv.org/abs/1208.3395.

[11] R. Škrekovski, List improper coloring of planar graphs. *Comb. Prob. Comp.*, **8** (1999), 293299.

[12] R. Steinberg, The state of the three color problem. Quo Vadis, Graph Theory?, *Ann. Discrete Math.* **55** (1993), 211–248.

[13] B. Xu, On $(3,1)^*$-coloring of planar graphs, *SIAM J. Disc. Math.*, **23** (2008), 205–220.

[14] L. Xu, Z. Miao, and Y. Wang, Every planar graph with cycles of length neither 4 nor 5 is $(1,1,0)$-colorable, *J Comb. Optim.*, DOI 10.1007/s10878-012-9586-4.

[15] L. Xu and Y. Wang, Improper colorability of planar graphs with cycles of length neither 4 nor 6 (in Chinese), *Sci Sin Math*, **43** (2013), 15-24. Doi: 10.1360/012012-70.

* Department of Mathematics, College of William and Mary, Williamsburg, VA, USA, 23185.

E-mail address: oshill@email.wm.edu, gyu@wm.edu

† Department of Mathematics, University of Rhode Island, Kingston, RI, USA, 02881.

E-mail address: diana@math.uri.edu

‡ Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, China, 321004.
E-mail address: lingjixu@126.com, yqwang@zjnu.edu