

4.1: Maximum and minimum values of functions

One of the most important applications of derivatives is on optimization problems, in which we are required to find the optimal (best) way of doing something. For example, the most efficient way, the cheapest way, the shortest time, the smallest cost et al. Those problems can be reduced to find the maximum or minimum values of functions. Let us first give a precise definition of maximum/minimum values of function.

1. Definition of the maximum/minimum values:

(1) A function f has a **global maximum** at c if $f(c) \geq f(x)$ for all x in the domain. The value of $f(c)$ is called the maximum value of f in the domain.

(2) A function f has a **global minimum** at c if $f(c) \leq f(x)$ for all x in the domain. The value $f(c)$ is called the minimum value of f in the domain.

The maximum and minimum values of f are called the **extreme values** of f .

(3) A function f has a **local maximum** at c if $f(c) \geq f(x)$ for all x near c .

(4) A function f has a **local minimum** at c if $f(c) \leq f(x)$ for all x near c .

From those definitions, we may see that when talking about maximum/minimum values, it is very important to know where x is.

Examples: find the global/local max/min values of the functions in the given domains.

(a) $f(x) = x^2$, $x \in [-1, 2]$ and $x \in R$

(b) $f(x) = x^3$, $x \in [-2, 2]$ and $x \in R$

From the examples, we see that global max/min values do not necessarily exist.

2. When does a function have maximum and minimum values?

The extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains a global maximum value $f(c)$ and a global minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Remark:

(i) The conditions of **continuous** and **on a closed interval** must be satisfied at the same time. Otherwise, we may not have global max/min values.

Question: Give an example such that f is not continuous and f has no global maximum and minimum values.

Question: Give an example such that f is continuous (but not on a closed interval) and f has no global maximum and minimum values.

(ii) This theorem is about the existence of global max/min values, and it says nothing about where the local max/min are.

Then what's the condition for the existence of local maximum/minimum values?

3. Fermat's Theorem and critical points

Fermat's Theorem: If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: Can you prove it?

Remarks:

(i) Fermat's Theorem suggests that to find the local maximum/minimum values of functions, we should look at points where $f'(x) = 0$ **OR** $f'(x)$ does not exist.

Critical points: a critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

(ii) $f'(c) = 0$ does not guarantee c is a local maximum or minimum value. For example: $f(x) = x^3$ at $x = 0$.

Example: Find the critical points of $f(x) = x^{3/5}(4 - x)$.

4. How to find the global maximum/minimum values of a function?

Now we are ready to find the global maximum/minimum values of a function on a closed interval.

Except at the endpoints of the interval, a global maximum/minimum value is also a local maximum/minimum value, thus we only need to consider endpoints and critical points to find the global maximum/minimum values.

Steps to find global max/min values: (**The closed interval method:**)

- (i) Find critical points of f in (a, b) , and find the function values at those points.
- (ii) Find the values of f at the endpoints, i.e., at a and b .
- (iii) The largest of (i) and (ii) is the maximum and smallest of (i) and (ii) is the minimum.

Example: Find the global maximum and minimum values of the functions in the given intervals.

(a) $f(x) = x^3 - 3x^2 + 1$ at $-1/2 \leq x \leq 4$.

(b) $f(x) = x - 2 \sin x$ at $0 \leq x \leq 2\pi$.

(c) $f(x) = 3x(x + 4)^{2/3}$ at $[-5, -1]$.