Math 412: Number Theory
Lecture 11 Möbius Inversion Formula

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Multiplicative functions

- Def: $\phi(n)$ is the number of elements in a reduced system of residues modulo n. (i.e., the number of coprimes to n in $\{1, 2, \ldots, n\}$.)
- Def: the sum of divisors of n: $\sigma(n) = \sum_{d|n} d$ and the number of divisors of n: $\tau(n) = \sum_{d|n} 1$
- For $f \in \{\phi, \sigma, \tau\}$, $f(mn) = f(m)f(n)$ if $(m, n) = 1$, namely, they are multiplicative functions.
- Thm: if f is a multiplicative function, then $F(n) = \sum_{d|n} f(d)$ is also multiplicative.
- And $\sum_{d|n} \phi(d) = n$.

- Let $F(n) = \sum_{d|n} f(d)$. Then what is f in terms of F?

$\underline{n=8}$: $F(8) = \sum_{d|8} f(d) = f(1) + f(2) + f(4) + f(8)$

$= F(1) + \ldots$

$F(1) = \sum_{d|1} f(d) = f(1)$

$F(2) = f(1) + f(2) \Rightarrow f(2) = F(2) - F(1)$

$F(4) = \underline{f(1) + f(2) + f(4)} \Rightarrow f(4) = F(4) - f(1) - f(2)$

$= F(4) - F(1) - (F(2) - F(1))$

$= F(4) - F(2)$.

$f(8) = F(8) - f(1) - f(2) - f(4) = F(8) - F(4) + 0 \cdot F(2)$

$f(n) = \sum_{d|n} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} F(d)$

Möbius Inversion

- Def: the Möbius function, $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1, n = 1 \\ (-1)^r, n = p_1 p_2 \ldots p_r \\ 0, \textit{otherwise} \end{cases}$$

$\mu(1) = 1$

$p_i \neq p_j$

$\mu(1) = 1$, $\mu(2) = (-1)^1 = -1$, $\mu(3) = (-1)^1 = -1$ $\mu(p) = -1$.

$\mu(4) = \mu(2^2) = 0$ $\mu(6) = \mu(2 \cdot 3) = (-1)^2 = 1$

$\mu(33) = \mu(3 \cdot 11) = (-1)^2 = 1$

Möbius Inversion

- Def: the Möbius function, $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1, n = 1 \\ (-1)^{r}, n = p_1 p_2 \dots p_r \\ 0, otherwise \end{cases}$$

- Thm: $\mu(n)$ is multiplicative.

Pf: $(m,n)=1$

$\mu(mn)=0$ if m has p^r with $r>1$ as a factor.

& in this case, $\mu(m)=0$. $\Rightarrow \mu(mn)=\mu(m)\mu(n)$

$m=p_1\cdots p_r$
$n=q_1\cdots q_t$ $\Rightarrow mn=p_1\cdots p_r q_1\cdots q_t$

$\mu(mn)=(-1)^{r+t}$

$\mu(m)=(-1)^r$, $\mu(n)=(-1)^t$ $\Big\} \Uparrow$

Möbius Inversion

- Def: the Möbius function, $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1, n = 1 \\ (-1)^r, n = p_1 p_2 \dots p_r \\ 0, \textit{otherwise} \end{cases}$$

$\mu(n)$ is multiplicative $\Downarrow$ $\sum_{d|n} \mu(d)$ is also multiplicative

- Thm: $\mu(n)$ is multiplicative.
- Thm: $\sum_{d|n} \mu(d) = 1$ if and only if $n = 1$, otherwise, it is 0.

"$\Leftarrow$": $n=1$. $\sum_{d|1} \mu(d) = \mu(1) = 1$.

"$\Rightarrow$": $\underline{n \neq 1}$: $n = p_1^{r_1} \cdots p_t^{r_t}$

$$\sum_{d|n} \mu(d) = \prod_{i=1}^{t} \left(\sum_{d|p_i^{r_i}} \mu(d) \right) = \prod_{i=1}^{t} \left(\mu(1) + \mu(p_i) + \mu(p_i^2) + \dots \right)$$

$$= \prod_{i=1}^{t} \left(1 + (-1) + 0 + \dots + 0 \right) = 0$$

Möbius Inversion

- Mobius Inversion Formula:

$$\text{If } F(n) = \sum_{d|n} f(d), \text{ then } f(n) = \sum_{d|n} \mu(d) F(n/d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d).$$

pf:

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n}\left[\mu(d) \cdot \sum_{e|\frac{n}{d}} f(e)\right] = \sum_{d|n}\left(\sum_{e|\frac{n}{d}} \mu(d) f(e)\right)$$

$$= \sum_{e|n} \sum_{d|\frac{n}{e}} \mu(d) f(e) = \sum_{e|n}\left(f(e) \cdot \sum_{d|\frac{n}{e}} \mu(d)\right)$$

$e \mid \frac{n}{d} \Leftrightarrow d \cdot e \mid n$

$\sum_{d|\frac{n}{e}} \mu(d) = \begin{cases} 1, & \frac{n}{e} = 1 \\ 0, & \text{o/w}. \end{cases}$

$$= f(1) \cdot 0 + \cdots 0 + f(n) \cdot 1 = f(n).$$

Applications of Möbius Inversion

$F(n)=n$

- From $n = \sum_{d|n} \phi(d)$, we have $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$. $= \left(\sum_{d|n} \frac{\mu(d)}{d}\right) n$

$\Downarrow$

$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot d$

$\Downarrow$

$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.

Applications of Möbius Inversion

- From $n = \sum_{d|n} \phi(n)$, we have $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$.

- From $\sigma(n) = \sum_{d|n} d$ we have $n = \sum_{d|n} \mu(n/d)\sigma(d)$.

$\underline{\underline{n=10}}$:

$$10 = \mu(10)\sigma(1) + \mu(5)\sigma(2) + \mu(2)\sigma(5) + \mu(1)\sigma(10)$$
$$= \sigma(1) - \sigma(2) - \sigma(5) + \sigma(10)$$

Applications of Möbius Inversion

- From $n = \sum_{d|n} \phi(n)$, we have $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$.
- From $\sigma(n) = \sum_{d|n} d$ we have $n = \sum_{d|n} \mu(n/d)\sigma(d)$.
- From $\tau(n) = \sum_{d|n} 1$ we have $1 = \sum_{d|n} \mu(n/d)\tau(d)$.

$\underline{n=10}$: $1 = \tau(10) - \tau(5) - \tau(2) + \tau(1)$

Applications of Möbius Inversion

- From $n = \sum_{d|n} \phi(n)$, we have $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$.
- From $\sigma(n) = \sum_{d|n} d$ we have $n = \sum_{d|n} \mu(n/d)\sigma(d)$.
- From $\tau(n) = \sum_{d|n} 1$ we have $1 = \sum_{d|n} \mu(n/d)\tau(d)$.
- Thm: if $F(n) = \sum_{d|n} f(d)$ and F is multiplicative, then f is also multiplicative.

Pf: From MIF, $f(n) = \sum_{d|n} \underline{\underline{\mu(\frac{n}{d})F(d)}}$ is multiplicative.

Converse of MIF:

- Thm: if $f(n) = \sum_{d|n} \mu(d) F(n/d)$, then $F(n) = \sum_{d|n} f(d)$.

$\frac{d}{e} = t$

pf:

$$\sum_{d|n} f(d) = \sum_{d|n} \sum_{e|d} \mu(e) F\left(\frac{d}{e}\right) = \sum_{d|n} \sum_{t|d} \mu\left(\frac{d}{t}\right) F(t)$$

$\frac{d}{t} = s$

$st = d | n$

$stx = n$

$sx = \frac{n}{t}$

$s | \frac{n}{t}$

$$= \sum_{t|n} \sum_{s|\frac{n}{t}} \mu(s) F(t)$$

$$= \sum_{t|n} \left[F(t) \sum_{s|\frac{n}{t}} \mu(s) \right] = F(n) \cdot 1 + F(\;) \cdot 0 + \cdots 0 + F(1) 0 = F(n).$$

$\sum_{s|\frac{n}{t}} \mu(s) = \begin{cases} 1, & \frac{n}{t} = 1 \\ 0, & o/w \end{cases}$

- Thm: if $f(n) = \sum_{d|n} \mu(d) F(n/d)$, then $F(n) = \sum_{d|n} f(d)$.

- In general, one can have Dirichlet product (or Dirichlet Convolution):

$$h * f = \sum_{n=d_1 d_2} h(d_1) f(d_2). = \sum_{d|n} h(d) f\left(\frac{n}{d}\right)$$

Then we have $g = \mu * f$ if and only if $f = 1 * g$.

$\Big\Downarrow$ let $h = \mu$

$g = \mu * f \iff f = 1 * g$

$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$