

1.3 EXERCISES

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ 2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy -graph: $\mathbf{u}, \mathbf{v}, -\mathbf{v}, -2\mathbf{v}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}$, and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$, and $-\mathbf{v}$.

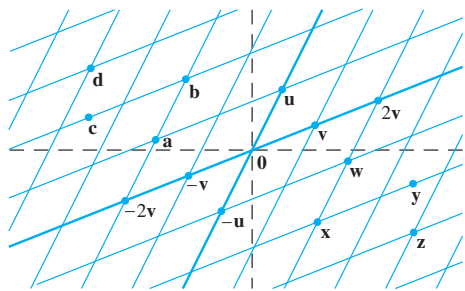
3. \mathbf{u} and \mathbf{v} as in Exercise 1 4. \mathbf{u} and \mathbf{v} as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

6. $x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of \mathbf{u} and \mathbf{v} . Is every vector in \mathbb{R}^2 a linear combination of \mathbf{u} and \mathbf{v} ?



7. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and \mathbf{d}

8. Vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and \mathbf{z}

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. $x_2 + 5x_3 = 0$ 10. $3x_1 - 2x_2 + 4x_3 = 3$
 $4x_1 + 6x_2 - x_3 = 0$ $-2x_1 - 7x_2 + 5x_3 = 1$
 $-x_1 + 3x_2 - 8x_3 = 0$ $5x_1 + 4x_2 - 3x_3 = 2$

In Exercises 11 and 12, determine if \mathbf{b} is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 .

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 13 and 14, determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

15. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?

16. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$. For what value(s) of h is \mathbf{y} in the plane generated by \mathbf{v}_1 and \mathbf{v}_2 ?

In Exercises 17 and 18, list five vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on \mathbf{v}_1 and \mathbf{v}_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

17. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

18. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

19. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 18.

21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ for all h and k .

22. Construct a 3×3 matrix A , with nonzero entries, and a vector \mathbf{b} in \mathbb{R}^3 such that \mathbf{b} is not in the set spanned by the columns of A .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \ 3]$.

b. The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.

c. An example of a linear combination of vectors \mathbf{v}_1 and \mathbf{v}_2 is the vector $\frac{1}{2}\mathbf{v}_1$.

- d. The solution set of the linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.
- e. The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.
24. a. When \mathbf{u} and \mathbf{v} are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains only the line through \mathbf{u} and the origin, and the line through \mathbf{v} and the origin.
- b. Any list of five real numbers is a vector in \mathbb{R}^5 .
- c. Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ has a solution amounts to asking whether \mathbf{b} is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
- d. The vector \mathbf{v} results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector \mathbf{v} .
- e. The weights c_1, \dots, c_p in a linear combination $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ cannot all be zero.
25. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
- a. Is \mathbf{b} in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
- b. Is \mathbf{b} in W ? How many vectors are in W ?
- c. Show that \mathbf{a}_1 is in W . [Hint: Row operations are unnecessary.]
26. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A .
- a. Is \mathbf{b} in W ?
- b. Show that the second column of A is in W .
27. A mining company has two mines. One day's operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kilograms of silver, while one day's operation at mine #2 produces ore that contains 40 metric tons of copper and 380 kilograms of silver. Let $\mathbf{v}_1 = \begin{bmatrix} 30 \\ 600 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 40 \\ 380 \end{bmatrix}$. Then \mathbf{v}_1 and \mathbf{v}_2 represent the "output per day" of mine #1 and mine #2, respectively.
- a. What physical interpretation can be given to the vector $5\mathbf{v}_1$?
- b. Suppose the company operates mine #1 for x_1 days and mine #2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kilograms of silver. Do not solve the equation.
- c. [M] Solve the equation in (b).
28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For

each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

- a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
- b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
- c. [M] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.
29. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be points in \mathbb{R}^3 and suppose that for $j = 1, \dots, k$ an object with mass m_j is located at point \mathbf{v}_j . Physicists call such objects *point masses*. The total mass of the system of point masses is

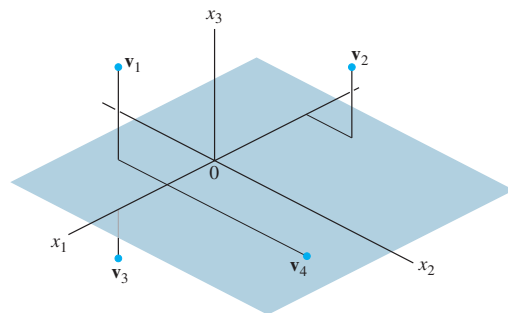
$$m = m_1 + \dots + m_k$$

The *center of gravity* (or *center of mass*) of the system is

$$\bar{\mathbf{v}} = \frac{1}{m}[m_1\mathbf{v}_1 + \dots + m_k\mathbf{v}_k]$$

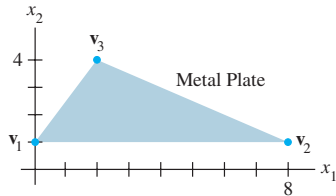
Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (2, -2, 4)$	4 g
$\mathbf{v}_2 = (-4, 2, 3)$	2 g
$\mathbf{v}_3 = (4, 0, -2)$	3 g
$\mathbf{v}_4 = (1, -6, 0)$	5 g



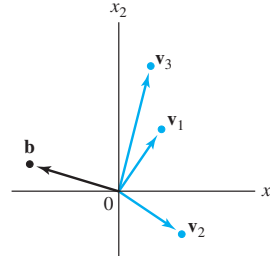
30. Let \mathbf{v} be the center of mass of a system of point masses located at $\mathbf{v}_1, \dots, \mathbf{v}_k$ as in Exercise 29. Is \mathbf{v} in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$? Explain.

31. A thin triangular plate of uniform density and thickness has vertices at $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (8, 1)$, and $\mathbf{v}_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.



- Find the (x, y) -coordinates of the center of mass of the plate. This “balance point” of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.
 - Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [Hint: Let w_1, w_2 , and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]
32. Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and \mathbf{b} in \mathbb{R}^2 , shown in the figure. Does the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ have a

solution? Is the solution unique? Use the figure to explain your answers.



33. Use the vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\mathbf{w} = (w_1, \dots, w_n)$ to verify the following algebraic properties of \mathbb{R}^n .
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
 - $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for each scalar c
34. Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of \mathbb{R}^n .
- $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
 - $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

SOLUTIONS TO PRACTICE PROBLEMS

1. Take arbitrary vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in \mathbb{R}^n , and compute

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\ &= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\ &= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition} \end{aligned}$$

2. The vector \mathbf{y} belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

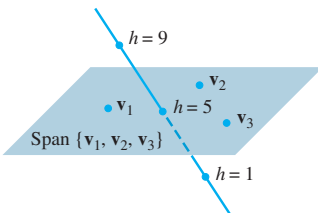
$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So \mathbf{y} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.



The points $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ lie on a line that intersects the plane when $h = 5$.