Math 432 – Combinatorics
Homework 2
Due Feb 5, 2016.

Solve the following problems. Show all your work. Each problem is 4 points.

1. Let p be a prime number. Prove that $p|\binom{p}{k}$ for integer k with $1 \leq k \leq p-1$. Use this to prove that $(1+x)^p \equiv 1 + x^p \mod p$.

2. Count the positive integer solutions to $\sum_{i=1}^{n} x_i \leq k$.

3. Count 4-words formed from the letters of COMBINATORICS.

4. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Show that $A^{n+1} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ for $n \geq 1$, where F_n is the n-th Fibonacci number.

5. Let D_n be the number of permutations of $[n]$ whose square is the identity permutation. Prove that $D_n = D_{n-1} + (n-1)D_{n-2}$ for $n \geq 3$ and $D_1 = 1, D_2 = 2$.

6. Problems on Catalan number. Let C_n be the n-th Catalan number.
   (a) A ballot path of length $2n$ is lattice path from $(0,0)$ to (n,n) that never rises above $y = x$. Show that there are C_n ballot paths of length $2n$.
   (b) A ballot list of length $2n$ is a list of n 1s and n 0s such that every initial segment has at least as many 1s as 0s. Show that there are C_n ballot lists of length $2n$.