Math 432 Lec 02 Counting principles, and binomial coefficient

- **Sum principle (counting by cases):** If a finite set A is partitioned into sets $B_1, B_2, \ldots, B_k$, then $|A| = \sum_{i=1}^{k} |B_i|$.
- **Product principle (counting by steps):** If the elements of A are build via (independent) successive choices, then $|A|$ is the product of the numbers of the options for the successive choices.

  Ex: How many ways to insert 3 letters to 4 mailboxes? How many functions from X to Y? How many injections from X to Y?
- **Principle of counting two ways:** when two formulas count the same set, their values are equal.

  Ex: $\sum_{i=1}^{n-1} i = n(n-1)/2$: count the pairs of elements in $[n]$, left by larger elements, and right by steps
- **Bijection Principle:** If there is a bijection from one set to another, then the two sets have the same size.

  Ex: the numbers of 0,1-lists of length n equals the number of subsets of $[n]$.

Examples:

(1) Let $\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}$. Show that there are $\binom{n}{k}$ k-subsets of $[n]$.

In the future, we will associate $\binom{n}{k}$ with the k-subsets of $[n]$.

(2) Properties of $\binom{n}{k}$:
  - (a) $\binom{n}{k} = \binom{n}{n-k}$ (bijection)
  - (b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. (count in two ways)
  - (c) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ (count in two ways, count by cases)

(3) Binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

From this formula we can draw the Pascal Triangle.

(4) Counting in poker: an ordinary deck has 52 distinct card. They are grouped into four suits, each with 13 ranked cards.
  - (a) how many distinct hands with five cards?
    $\binom{52}{5} = 259860$.
  - (b) how many "two pairs"?
    $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$ (we do this in four steps: first choose 2 ranks as pairs, then choose two suites from each of the two chosen ranks, the last step is to choose the fifth card.)
  - (c) how many "flush"? (a flush consist of five cards in one suit)
    $4\binom{13}{5}$ (we do this in two steps: first choose a suit among four suits, then choose 5 cards in the chosen suit.)
  - (d) how many "straights"? (a straight consists of one card each in five consecutive ranks, except the "ace" can be considered either the lowest or the highest rank)
    $10\cdot 4^5$ (we do this in five steps: first there are 10 possible ranks to start a straight, and for each of the five cards, there are 4 suits to choose.)